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A PROBLEM IN ELEMENTARY GEOMETRY.

BY KURT MAHLER

RECENTLY, in connection with some work on Diophantine approximations, I encountered the following problem on triangles.

Let T be a triangle with vertices A, B, C which are, respectively, inner points of the sides a, b, c of a second triangle t. Is it always possible to move T into a new position where its vertices are inner points of t?

I give here an affirmative answer to the problem and prove, moreover, that it suffices to apply to T an arbitrarily small rotation about a suitably chosen point of the plane. I am indebted to my Manchester colleagues for a number of simplifications of this solution, arrived at when discussing the problem with them.

In the proof, several cases will be distinguished. We always denote by α, β, γ the perpendiculars to the lines a, b, c at A, B, and C, respectively ; these perpendiculars will be thought of as extending to infinity in both directions.

Case 1. α and β intersect at a point D which does not also lie on γ (Fig. 1 and 2).

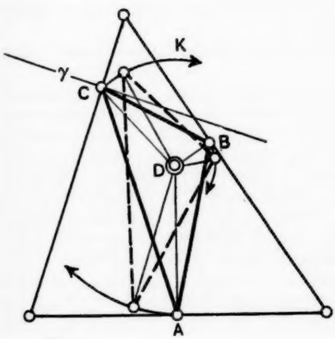

Fig. 1

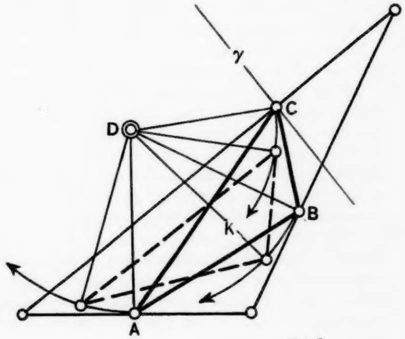

Fig. 2

Q

If T is rotated about D by an arbitrarily small angle, C describes a small arc of the circle K with centre at D. The side c of t is not a tangent to K; hence the direction of the rotation can be chosen such that the new position of C lies inside t. This rotation has thus the required property since it obviously transports also A and B into inner points of t.

Case 2 : All three lines α, β, γ intersect at a point D in the interior of t (Fig. 3).

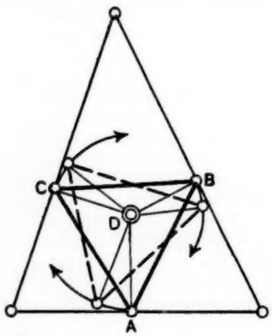

Fig. 3

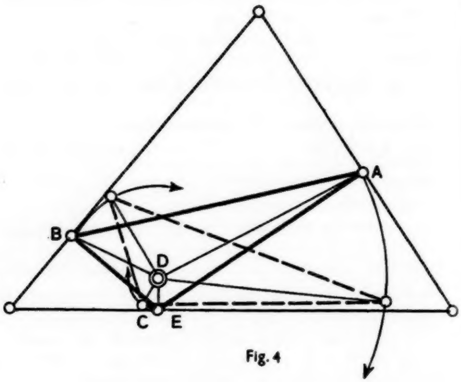

Fig. 4

Evidently every sufficiently small rotation of T about D has the required property.

Case 3 : The three lines α, β, γ intersect at a point E which lies, say, on the side c of t (Fig. 4).

The hypothesis implies that E coincides with C, and that α is the side AC and β the side BC of T. Select a point D on γ arbitrarily near to E and inside T. The smaller angle at A of AD with the side a is less than, but arbitrarily near to, 90°. Hence there is a rotation of T about D such that A, after first leaving t, is changed into an inner point of t, and that both B and C become likewise inner points of t. Moreover, the angle of this rotation can be made arbitrarily small by taking D sufficiently near to $C=E$.

Case 4 : The three lines α, β, γ intersect at a point E which lies outside t, say on the outer perpendicular γ (Fig. 5).

Draw the circle, K say, that passes through C and E and has as its centre the midpoint of the line segment CE; further select a point D arbitrarily near to E on K. Evidently K touches the side c of t on the outside of the triangle. If P is any point on K which does not lie on the small arc DE of this circle, the angle $\angle DPE$ has the constant value $\angle DCE$, $=\zeta$ say. Further the points A and B are separated from D and E by the tangent c of K. It follows therefore that the two angles $\angle DAE$, $=\xi$ say, and $\angle DBE$, $=\eta$ say, are both smaller than ζ. We can then select an angle ϕ which is smaller than 2ζ, but greater than both 2ξ and 2η; moreover, ϕ will be arbitrarily small if D was chosen sufficiently near to E. We rotate now T about D by the angle ϕ in such a direction that A and B first leave t and afterwards become inner points of t; then, at the same time, C has likewise been changed into an inner point of t. This concludes the proof.

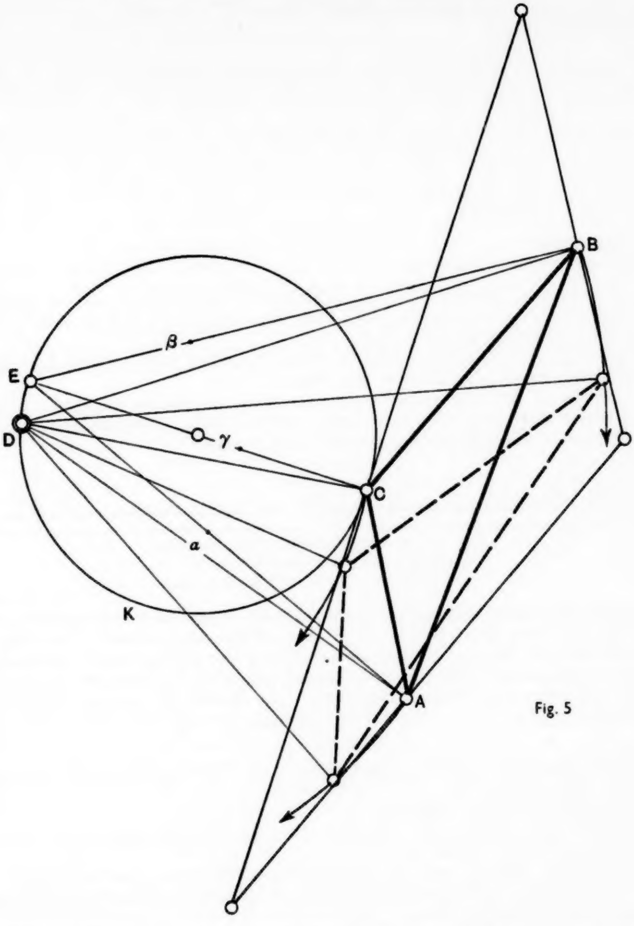

Fig. 5

It may be noted that the theorem has no obvious generalization to polygons of more than three sides. Thus there exist rectangles with their vertices on the sides of a square Q that cannot be moved into any new position where their vertices are inner points of Q.

The theorem may be extended to simplices in more dimensions.

K.M.

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A MULTI-PURPOSE VISUAL AID

By G. R. Clark.

The purpose of this article is to describe some apparatus which is easy to construct from materials that are reasonably cheap, and which has been found useful as a visual aid in many branches of elementary mathematics. It is thought that this apparatus will satisfy the need to which attention is drawn in § 4.8 of the *Trigonometry Report*, although it was being used before the report was published. After the apparatus has been described, some of its applications will be briefly indicated and a more detailed account of one application will follow.

The apparatus

The apparatus consists of:

1. A frame, of three-ply (21 inches long by $13\frac{1}{2}$ inches wide) to which are nailed pieces of wood (1 inch square) having a groove $\frac{3}{16}$ inch wide and $\frac{1}{4}$ inch deep along one face (see Fig. 1);
2. A sheet of perspex (18 inches long, 12 inches wide and $\frac{3}{16}$ inch thick) which slides into the grooves in the frame;
3. Sheets of drawing paper (approximately 24 inches long by 12 inches wide) which slide between the perspex and the back of the frame;
4. Pieces of wire, of varying lengths and shapes.

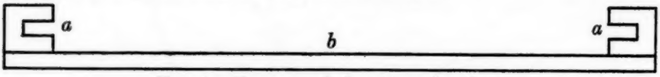

Fig. 1. Frame, end elevation, showing:
(*a*) grooves for perspex sheet;
(*b*) space for sheets of drawing paper.

Two frames have been prepared, each with a pair of screw-eyes by which they can be hung from hooks at the top of the blackboard. One frame is mounted so that its length is horizontal; the other has its length vertical, and a piece of three-ply nailed to the bottom prevents the drawing paper from sliding out when the frame is mounted. The frames are painted light blue.

Five sheets of perspex, with a different curve on each, are used. The curves are:

1. A straight line, passing through the centre of the sheet and making an angle of 45° with the edge;
2. A circle, of radius 5 inches;
3. A parabola, which passes through the origin and the points (± 5, 15), the scale on each axis being in inches;
4. An ellipse, whose axes are approximately 15 inches and 10 inches;
5. A sine-curve, of amplitude 5 inches, a little more than three complete periods being shown.

These curves are painted on the perspex in red enamel. Obtaining satisfactory curves without spoiling the perspex presented a problem that was, however, easily solved. Each sheet of perspex was supplied with a sheet of paper glued to one face. The curve was drawn on this paper, having first been plotted on graph paper if necessary. The paper was then cut carefully, with a razor blade, close to the curve, so that a strip approximately $\frac{1}{4}$ inch wide bounded the original curve. This strip was removed, and the enamel was applied to the perspex while the rest of the paper was firmly attached. When the enamel was dry, the paper was removed, leaving a curve with clearly defined edges.

Several sheets of drawing paper have been prepared with axes in varying positions and with different scales. Each axis and scale is clearly labelled.

The sheets of perspex are interchangeable : any curve may thus be mounted in either frame. Several backgrounds may be placed behind the perspex, and as the paper is easily withdrawn, one curve may play several different roles in a few minutes.

At the centre of the circle, a hole has been drilled through the perspex and a bolt (approximately 1 inch long and $\frac{1}{4}$ inch in diameter) is permanently fixed through the hole. To the bolt may be attached some of the pieces of wire which have been coiled at one end so that they fit closely round the bolt. A wing-nut serves either to keep the wire in position or to allow it to rotate. One such wire is 10 inches long. Another (see Fig. 2) forms a radius AB, and has been bent so that 1 inch BC is perpendicular to the perspex when it is mounted. Other wires have been prepared to fit on BC : one is 6 inches long, is coiled at one end, while the other end is weighted so that the wire hangs vertically ; another is 12 inches long and has been coiled into a small circle at E, and the weight at F ensures that the wire hangs vertically when the coil is fitted to BC. These latter wires are green, while those which fit over the bolt are yellow.

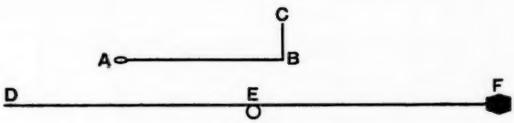

FIG. 2.
Wires to be used with the circle.

Applications

Many ways in which the apparatus may be used will doubtless suggest themselves. Some are given below.

1. *Algebra*. Illustration of (A) direct variation, and (B) the method of finding graphically the minimum or maximum value of the function

$$y = ax^2 + bx + c.$$

For (B), points have been plotted on drawing paper : (i), in which each value of y, except the minimum, occurs twice ; (ii), in which each value of y occurs twice, the minimum not appearing in the table of values ; (iii), in which no value of y occurs twice. When these cases are discussed in class the general shape of the curve is apparent. The sheets of paper are placed in turn behind the parabola, and the minimum value so obtained is compared with that obtained by the pupils from their graph books. In this way it becomes obvious that, for cases (ii) and (iii), it is necessary to take another value for x (preferably half-way between those which give the two lowest values of y) in order to find accurately the minimum value of y.

2. *Analytical geometry*. Illustration of (A) gradient and y-intercept for the straight line, and (B) the effect of change of origin on the equations of straight line, parabola and circle.

3. *Calculus*. Demonstration of the " limiting process " : a straight piece of wire is used as a chord of the parabola. As δx decreases, so does δy, and the gradient of the chord through the point under consideration approaches a definite limit, which is seen to be the gradient of the tangent at that point, for the chord itself passes, in the limit, to the tangent. (For such work as this, it is advisable to place the perspex in the frame so that the curve is downwards ; then the sliding wire does not scratch the paint from the perspex.)

4. *Trigonometry.*

(A) The dependence on θ of tan θ, sin θ, cos θ may be shown clearly by using an appropriate " back-drop " and attaching wires to the bolt at the centre of the circle.

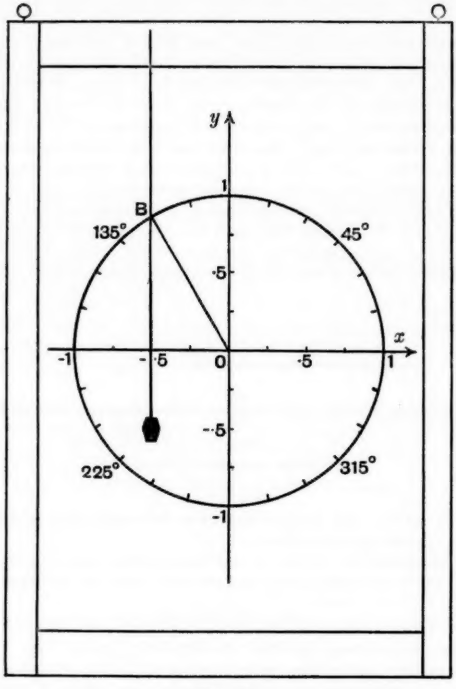

FIG. 3.

Circle on perspex in frame, with wires showing cos 120°. All figures, etc., are on drawing paper behind perspex.

(i) For tan θ, we use the 10 inch wire. The back-drop consists of a tangent to the circle at the point (1,0), graduated with the radius of the circle as a unit, and angles (30°, etc., or $\pi/6$, etc., according to the stage of development of the class) are marked around the circumference of the circle. Interesting features are: the name of the ratio is linked with the tangent to the circle; the increase of tan θ with θ becomes apparent, including the fact that the ratio increases very rapidly as θ approaches 90°; and this method stimulates abler pupils to find out what happens to the ratio when θ is greater than 90°, a problem which they can solve for themselves by using the apparatus.

(ii) For the other two ratios, the wire ABC of Fig. 2, having the shorter of the two weighted wires attached to BC, is used with a back-drop consisting of axes for x and y with the origin at the centre of the circle. The axes are graduated with the radius of the circle as the unit, and angles are marked

around the circumference of the circle. Further details of this will be given later.

(B) The sine curve may be used in a variety of ways: it immediately becomes the cosine curve when the back-ground is moved a quarter of a period to the right; it represents $\sin(x+\alpha)$, or $\sin(nx+\alpha)$ by using a back-drop with a different scale on the x-axis; different axes produce the curves for $\sin^2 x$, $\cos^2 x$, while interchanging the axes of x and y gives the curves for arc $\sin x$, arc $\cos x$.

(C) Both frames may be used together to develop the graphs of the sine and cosine functions for angles of any magnitude. The method of development for the cosine will be described, because the apparatus lends itself more readily to that function, although a modified apparatus which is equally applicable to either function has been described by A. W. Young in *The Australian Mathematics Teacher*, V, No. 2 (July, 1951). Previous work includes: (i) definition of ratios of acute angles, in terms of the sides of a right-angled triangle; (ii) graphs of the ratios of acute angles; (iii) some time later the definitions are re-stated in terms of the coordinates of a point whose locus is a circle with its centre at the origin; this leads to the idea that the point considered need not lie in the first quadrant, that the angle between the x-axis and the radius vector need not be acute.

The apparatus used consists of the circle with the wires ABC and DEF of Fig. 2, together with the back-drop described under application 4 (A) (ii) above. (See Fig. 3). As the point B moves from coincidence with Ox to coincidence with Oy, the wire EF indicates on Ox the value of $\cos\theta$, where $\theta=\angle xOB$. (The wire DE becomes the indicator when θ is a reflex angle). This is summed up in the graph of $z=\cos\theta$, which is then displayed, having a sheet of drawing paper placed on top of the perspex so that only the first quarter-period of the curve appears. (See Fig. 4).

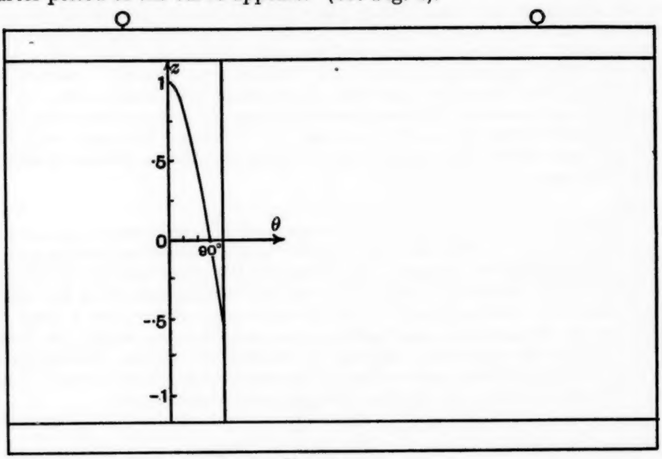

FIG. 4.

Cosine curve on perspex in frame, with covering sheet of drawing paper moved to the right to show cos 120°.

There are then two independent ways of obtaining the extension to non-acute angles.

(*a*) When B moves beyond the y-axis, what is the x-coordinate of B? What does this tell you about $\cos\theta$?

(*b*) If θ increases beyond 90°, what do you expect will happen to the curve $z=\cos\theta$? Does the shape of the curve lead you to expect that it will bend up. or that it will continue downwards?

Answers to these questions are examined in turn, using the apparatus, For (*b*) we move the covering paper slightly to the right, and we see that one answer confirms the other. Proceeding in a series of jerks—moving B, say, 15° and then moving the covering sheet through the same amount—the graph of $z=\cos\theta$ is built up. It is advisable at each critical angle (the multiples of 90°) to stress both approaches to what happens next, varying the order so that sometimes (*b*) is taken before (*a*).

The next stage is to plot the curve. For this, it is not sufficient to know that "when θ is obtuse, $\cos\theta$ is negative". General statements such as "when θ increases from 0° to 90°, $\cos\theta$ decreases from 1 to 0" and "when θ increases from 90° to 180°, $\cos\theta$ decreases from 0 to -1" are obvious from both pieces of apparatus, and they lead to the discovery of some such generalisation as "$\cos(180°-\theta)=-\cos\theta$, where θ is an acute angle". This last generalisation can also be derived readily from the apparatus; for example, cos 120° is obviously negative, but what is its actual value? Angles which are allied to 120° are 30° and 60°, and it is easy to demonstrate that the magnitude of cos 30° is much larger than that of cos 120°, while cos 60° is equal in magnitude to cos 120°. Repeating this argument for each of the 15° jerks of the previous paragraph helps to drive home what may be termed the "180°-symmetry" for angles up to 270°; and this suggests symmetry about 360° for the next half-period of the graph. Further, the sceptical student has this advantage, that he can move the apparatus into his "problem position" and then try to figure it out for himself.

Finally, the curve is committed to memory. When its general shape and characteristics are known thoroughly, it gives a more reliable clue to the value of, say, cos 235°, than the multitude of formulae of the type $\cos(180°+\theta)=-\cos\theta$, $\cos(270°-\phi)=-\sin\phi$, etc. In "ten-minute terrors" (twenty to thirty questions on formulae and their applications are given orally, to be completed in ten minutes) or their smaller brothers, "five-minute furies", success comes far more frequently to those who trace out the curve with the tip of their finger (or of their nose!) than to those who try to find the formula which fits the case.

Conclusion.

It is hoped that those who have the energy, and take the time, to construct this apparatus, will find that it is far superior to the "protractor diagram"; the pupil does not need to imagine, for he can see the point moving round the circle and also the changing values of x and y; the apparatus is far more tractable than the film suggested in the *Trigonometry Report*, for it may be quickly used to demonstrate the value of the ratios of any angle; it is far more lively than the pedestrian manner of plotting the graph. In addition, this is but one of the many applications of the apparatus; it is valuable in all graphical work, as well as in algebra, calculus and trigonometry.

G. R. C.

GLEANINGS FAR AND NEAR.

1794. All mathematicians, I agree, are subject to schizophrenia, being apt to brood on trilinear co-ordinates when they ought to be thinking about life.—Harold Nicolson, in *The Observer*, February 7, 1954.

CONVERSION OF VARIATION PROBLEMS INTO ISOPERIMETRICAL PROBLEMS

By K. E. Bullen.

The purpose of this article is to apply a method, previously discussed* in some detail, to the solving of standard elementary problems in the Calculus of Variations and to show how rapidly the solutions can be derived by introducing (p, r) and certain other coordinates. In each case a Lagrange undetermined multiplier is introduced in the course of the solution, so that the problem if not already an isoperimetrical problem is converted to one.

1. *Shortest curve joining two fixed coplanar points A and B, using* (p, r) *coordinates.* In usual notation, the integral to be made stationary is $\int ds$, i.e. $\int r(r^2-p^2)^{-\frac{1}{2}}\,dr$. At the terminals, r is assigned while p is not. So (cf. the extended solution of the previous article), it is necessary to set down the conditional equation

$$\alpha = \int pr^{-1}(r^2-p^2)^{-\frac{1}{2}}\,dr,$$

where α is the (assigned) angle subtended at the origin by AB. The shortest curve is then obtained by setting $\partial\eta/\partial p = 0$, where

$$\eta = r(r^2-p^2)^{-\frac{1}{2}} - \lambda pr^{-1}(r^2-p^2)^{-\frac{1}{2}},$$

and λ is an undetermined multiplier. The result on simplifying is $p=\lambda$, the equation of a straight line.

2. *Shortest curve joining two fixed coplanar points, using the coordinates* (r, ϕ). The solution of the Problem 1 is simpler still if the coordinates are taken as r and ϕ, where ϕ is the angle between the tangent and radius vector to a point of the curve. Then

$$\int ds = \int \sec\phi\,dr$$

and

$$\alpha = \int r^{-1}\tan\phi\,dr,$$

and the shortest curve is given by

$$\frac{\partial}{\partial\phi}(\sec\phi - \lambda r^{-1}\tan\phi) = 0,$$

i.e. by $r\sin\phi = \lambda$,—again the equation of a straight line.

3. *The Brachistochrone.* In this problem the curve is required which gives the time of quickest descent under constant gravity between two fixed points

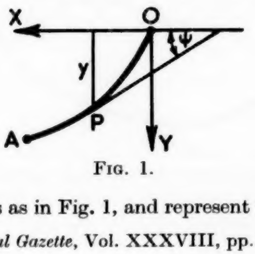

Fig. 1.

O and A, say. Take axes as in Fig. 1, and represent the position of a point P

* *Mathematical Gazette*, Vol. XXXVIII, pp. 172–4, 1954.

of the curve by (y, ψ). The velocity v at P is $\sqrt{(2gy)}$, and the integral to be made stationary, namely $\int ds/v$, is proportional to $\int y^{-\frac{1}{2}} \operatorname{cosec} \psi \, dy$. The conditional equation needed is

$$\int \cot \psi \, dy = \text{constant}, \quad \text{...(1)}$$

expressing the constancy of the horizontal projection of OA. The required curve is then given by

$$\frac{\partial}{\partial \psi}(y^{-\frac{1}{2}} \operatorname{cosec} \psi - \lambda \cot \psi) = 0,$$

i.e. by

$$y = \lambda^{-2} \cos^2 \psi, \quad \text{...(2)}$$

where λ is constant. The curve is thus part of a cycloid which in usual notation is given by $y = 2a \cos^2 \psi$.

4. *Surface of revolution of minimum area.* Again using (y, ψ), the area of the surface of revolution of the curve AB in Fig. 2 about OX, namely $\int 2\pi y \, ds$, is proportional to $\int y \operatorname{cosec} \psi \, dy$. This integral, which is to be minimised, is the same as in Problem 3 except that $y^{-\frac{1}{2}}$ is replaced by y. Also the same equation of condition (1) holds. The solution is therefore (2) with y replaced by y^{-2}, i.e. $y = \lambda \sec \psi$, the equation of a uniform catenary.

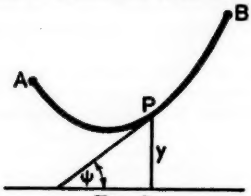

FIG. 2.

5. *Curve of maximum area, given the perimeter.* This problem is already isoperimetrical, being in fact the famous original isoperimetrical problem. In terms of (r, ϕ)—see Fig. 3—the area to be made a maximum, namely

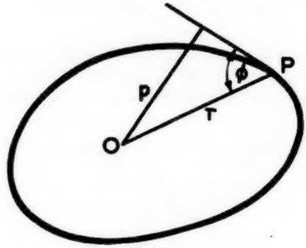

FIG. 3.

$\int p \, ds$, is equal to $\int r \tan \phi \, dr$, and the assigning of the perimeter gives

$$\text{constant} = \int ds = \int \sec \phi \, dr.$$

But with (r, ϕ) coordinates we need the second equation of condition

$$2\pi = \int d\theta = \int r^{-1} \tan \phi \, dr.$$

The sought curve is then given by

$$\frac{\partial}{\partial \phi}\left(r \tan \phi + \lambda \sec \phi + \frac{\mu}{r} \tan \phi\right) = 0,$$

where λ and μ are constants, i.e. by

$$r^2 + \lambda r \sin \phi + \mu = 0,$$

the equation of a circle.

6. *Uniform chain of given length hanging under constant gravity between two fixed points.* By the minimum energy principle, the potential energy, proportional to $\int y \, ds$, i.e. $\int y \operatorname{cosec} \psi \, dy$ (See Fig. 2), is to be made a minimum. Since the length $\int ds$ is fixed, we have $\int \operatorname{cosec} \psi \, dy = \text{constant}$. And we need the further condition

$$\text{constant} = \int dx = \int \cot \psi \, dy.$$

Hence the required curve is given by

$$\frac{\partial}{\partial \psi}(y \operatorname{cosec} \psi - \lambda \operatorname{cosec} \psi - \mu \cot \psi) = 0,$$

i.e.
$$y - \lambda = \mu \sec \psi,$$

a uniform catenary.

7. *Geodesic joining two points A and B on a spherical surface.* Let P and Q be two neighbouring points of a curve on the sphere joining A and B. The

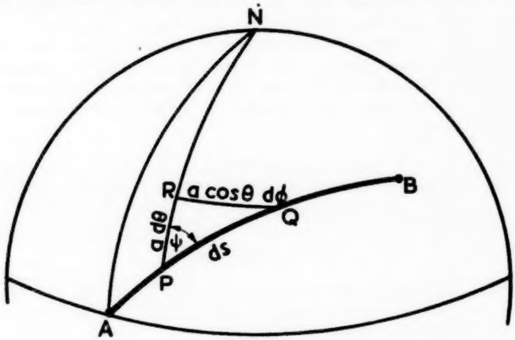

FIG. 4.

coordinates of P will be taken as the colatitude θ and longitude ϕ, the north pole N being taken so that A is on the equator. Then if PN is a great-circle arc, and R is taken on PN so that RQ is parallel to the equator, we see from Fig. 4 that

$$ds = a \sec \psi \, d\theta, \quad \sin \theta \, d\phi = \tan \psi \, d\theta,$$

where a is the radius and ψ is as shown. It is required to minimise the integral $\int ds = \int a \sec \psi \, d\theta$, subject to the condition that

$$\text{constant} = \int d\phi = \int \operatorname{cosec} \theta \tan \psi \, d\theta.$$

Hence we require

$$\frac{\partial}{\partial \psi}(\sec \psi - \lambda \operatorname{cosec} \theta \tan \psi) = 0,$$

where λ is constant, i.e.

$$\sin \psi \sin \theta = \text{constant} = \sin NAP, \tag{3}$$

where NA is a great-circle arc. Comparison of (3) with the formula for a spherical triangle ANP composed of great-circle arcs shows that AP and therefore the required geodesic must be a great-circle arc.

(As in the case of the previous note, the integrands in the above solutions may pass through infinite values. This happens when the independent variable is stationary at a point, K say, of the sought curve. The equation obtained for the curve then applies separately to the branches on either side of K; but the essential interpretation is not affected.)

K. E. B.

1795. The free world ought to realise that the newly freed non-white world sees the cold war in a different light from the way the democracies of the white world do. To the latter the cold war is a very simple linear equation. The West (democracy) versus the Iron Curtain (totalitarianism) equals good versus bad. To the average Indian, and Mr. Nehru too is an average Indian, the equation from linear becomes quadratic. The West (democracy plus colour bar) versus the Iron Curtain (totalitarianism minus colour bar) equals good plus bad versus bad plus good. The result is no longer a straightforward number but the complicated square root of good and evil, for to the non-white people freedom and equality are equally important.—*Manchester Guardian*. [Per Mr. Alec D. Walters.]

1796. Mr. C. H. Perkins recalled that when once genially twitted about the price of pies, Mr. Davis had the bright rejoinder : " I sell them below cost, and it is only the number I sell that enables me to make any profit."

(Mr. Davis was caretaker at the Boys' High School, Christchurch, N.Z.)—*The Press*, October 2, 1953. [Per Mrs. Dromgoole.]

1797. Sir,—In your issue of December 31 you quote Mr. B. S. Morris as saying that many people are disturbed that about half the children in the country are below the average in reading ability. This is only one of many similarly disturbing facts. About half the church steeples in the country are below average height ; about half our coal scuttles below average capacity, and about half our babies below average weight. The only remedy would seem to be to repeal the law of averages.

Yours faithfully,

ALAN STEWART.

—*The Times*, January, 1954. [Per Mr. E. H. Lockwood.]

PARABOLAS RELATED TO A TRIANGLE*

BY P. S. RAU.

(ANDHRA UNIVERSITY, WALTAIR, SOUTH INDIA)

LET ABC be a plane triangle with D, E, F the midpoints of the sides BC, CA, AB respectively. Let L, L' be two real intersecting straight lines in the plane. Then it can easily be seen, as in § 1 below, that if parallelograms are described on the three sides as diagonals, each with its sides parallel to L, L', then the other diagonals of these parallelograms are concurrent at a finite point of the plane, which may be denoted by $P(L, L')$. An attempt to determine those points of the plane which are such points of concurrence has led to the following results.

(A) The set of points $P(L, L')$ for varying real lines L, L' are the points D, E, F together with the open regions $\mathcal{D}'$, $\mathcal{E}'$, $\mathcal{F}'$ (shaded in Fig. 1) where $\mathcal{D}'$ is that open region, bounded by DE produced, DF produced and EF, which does not contain D, and $\mathcal{E}'$, $\mathcal{F}'$ are similar open regions.

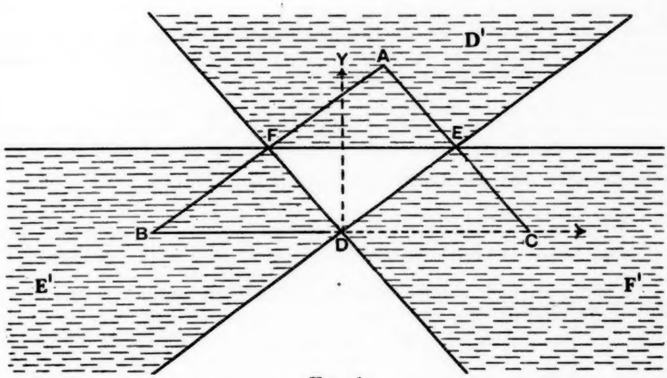

FIG. 1.

(B) If L is kept fixed, the set of points $P(L, L')$ constitutes the parabola through D, E, F, with its axis parallel to L. Then the set of points $P(L, L')$ for varying real L and L' can also be regarded as constituting all the real points of the real parabolas through D, E, F.

(C) If L and L' are at right angles, then $P(L, L')$ describes the nine-points circle of the triangle ABC.

1. Let DC and DY (perpendicular to DC) be taken as axes of coordinates. Then with no loss of generality the coordinates of A, B, C can be taken as (a, b), $(-1, 0)$ and $(1, 0)$ respectively, where $a \geqslant 0$, $b > 0$.

1.1. Let one of L, L' be parallel to $y = 0$, and the other parallel to $x - ly = 0$.

Then the equations of the pairs of lines parallel to L and L' through A, B, C are respectively given by

$$\begin{aligned} S_1 &\equiv \{(x-a) - l(y-b)\}(y-b) = 0, \\ S_2 &\equiv \{(x+1) - ly\}y = 0, \\ S_3 &\equiv \{(x-1) - ly\}y = 0. \end{aligned}$$

* The results of this paper were announced by me at the 17th Conference of the Indian Mathematical Society held at Bangalore, in December 1951.

Since the equations $S_2 - S_3 = 0$, $S_3 - S_1 = 0$, $S_1 - S_2 = 0$ are linear in x and y, they represent the other diagonals of the parallelograms on BC, CA, AB as diagonals respectively ; and these equations reduce to

$$\begin{aligned} D_1 &\equiv y = 0 \\ D_2 &\equiv y(a-1) + b(x-a) + b^2 l = 0, \\ D_3 &\equiv y(a+1) + b(x-a) + b^2 l = 0, \end{aligned}$$

showing that these diagonals are concurrent at the point $(a - bl, 0)$.

Also if any point $(x_1, 0)$ where $x_1 \neq 0$, is to be the point $P(L, L')$ for some L and L', then we find that the other diagonal of the parallelogram on BC as diagonal coincides with BC itself, so that the other two vertices of this parallelogram lie on BC, and therefore one of L, L' must be parallel to the line BC, that is, to $y = 0$. Further, x_1 satisfies the three equations $D_1 = 0$, $D_2 = 0$, $D_3 = 0$, and hence we have $x_1 - a + bl = 0$, showing that the other direction is unique and is parallel to $bx = (a - x_1)y$.

If, however, $x_1 = 0$, and one of L, L' is parallel to $y = 0$, then the other direction is parallel to $bx = ay$, which is the equation of the median through the midpoint of the side BC.

1.2. Neither of L, L' is parallel to $y = 0$.

Then their equations can be respectively taken as $x - ly = 0$, $x - l'y = 0$, where $l \neq l'$. As in 1.1, the equations of the pairs of lines parallel to L and L' through A, B, C are respectively given by

$$\begin{aligned} S_1 &\equiv (x-a)^2 + ll'(y-b)^2 - (l+l')(x-a)(y-b) = 0, \\ S_2 &\equiv (x+1)^2 + ll'y^2 - (l+l')(x+1)y = 0, \\ S_3 &\equiv (x-1)^2 + ll'y^2 - (l+l')(x-1)y = 0 ; \end{aligned}$$

and in the same way we obtain

$$\begin{aligned} D_1 &\equiv 2x - (l+l')y = 0, \\ D_2 &\equiv 2x(a-1) - (a^2-1) + ll'b(2y-b) - (l+l')\{y(a-1) + b(x-a)\} = 0, \\ D_3 &\equiv 2x(a+1) - (a^2-1) + ll'b(2y-b) - (l+l')\{y(a+1) + b(x-a)\} = 0, \end{aligned}$$

for the equations of the other diagonals of the parallelograms on BC, CA, AB as diagonals respectively.

Also
$$2D_1 + D_2 - D_3 \equiv 0 ;$$

and
$$\begin{vmatrix} 2 & -(l+l') \\ 2(a-1) - b(l+l') & -(a-1)(l+l') + 2bll' \end{vmatrix} = \begin{vmatrix} 2 & -(l+l') \\ -b(l+l') & 2bll' \end{vmatrix}$$
$$= -b(l-l')^2 \neq 0,$$

since b is the area of the triangle ABC, and $l \neq l'$. Hence $D_1 = 0$, $D_2 = 0$, $D_3 = 0$ are concurrent at a *finite* point of the plane.

2. Let any point (x_1, y_1) of the plane be the point $P(L, L')$ for some L and L'. We observe that if $y_1 = 0$, $x_1 \neq 0$, then one of L, L' must be parallel to $y = 0$, and the other direction is uniquely given. Also if $y_1 = 0$ and $x_1 = 0$, and one of L and L' is taken parallel to $y = 0$, then the other direction is that of the median through the midpoint of BC. Excluding these cases, we can assume that neither of L, L' is parallel to $y = 0$.

Let the equations of L and L' be, respectively,

$$x - ly = 0, \qquad x - l'y = 0.$$

Then x_1, y_1 satisfy the equations $D_1 = 0$, $D_2 = 0$, $D_3 = 0$ of 1.2, which are equivalent to

$$2x - (l+l')y = 0, \quad \text{(i)}$$
$$(2y-b)bll' - b(x-a)(l+l') = a^2 - 1. \quad \text{(ii)}$$

We thus obtain

$$b^2y^2(2y-b)^2(l-l')^2 = 4\{b^2x^2(2y-b)^2 - by(2y-b)[y(a^2-1)+2bx(x-a)]\}. \quad \ldots\text{(iii)}$$

2.1. $y_1 = 0$.

In this case, $x_1 = 0$ by (i), and from (ii) it follows that l and l' must satisfy the equation

$$(a-bl)(a-bl') = 1,$$

which holds only provided $l \neq a/b$ and $l' \neq a/b$. There is an infinity of pairs of values of l and l' satisfying this relation.

Thus if the midpoint of the side BC is to be the point $P(L, L')$ for some L and L', then L and L' are either parallel to the side BC and the median AD, or there is an infinity of pairs L, L', none of them being parallel to the side BC and the median AD, but one of the directions is uniquely determined by the other.

2.2. $y_1 = \frac{1}{2}b$.

Then (i) and (ii) give

$$b(l+l')(x_1-a) = 4x_1(x_1-a) = -a^2-1\,;$$

that is

$$\{x_1 - \tfrac{1}{2}(a+1)\}\{x_1 - \tfrac{1}{2}(a-1)\} = 0.$$

Hence the only points on the line EF which are $P(L, L')$ for some L and L' are the points E and F.

2.3. $y_1 \neq \frac{1}{2}b$, $y_1 \neq 0$.

Here l and l' will be real and distinct only if the right hand side of (iii) is positive, and this condition reduces to

$$(2y_1-b)\{y_1(a-1)-bx_1\}\{y_1(a+1)-bx_1\} < 0.$$

Hence (x_1, y_1) must be a point of the open regions $\mathcal{D}'$, $\mathcal{E}'$, $\mathcal{F}'$. The result (A) is thus proved.

Further, l and l' will be conjugate complex numbers if the preceding inequality is reversed. Hence the set of points $P(L, L')$ for varying conjugate complex lines L, L' are the interior points of the triangle DEF and the points of the three open regions bounded by: ED produced, FD produced; DE produced, FE produced; and DF produced, EF produced.

3. Let L be kept fixed, so that l is a constant. If $P(L, L')$ is the point (x, y) then x and y satisfy the equations (i) and (ii), which may be written

$$2x - (l+l')y = 0$$
$$b\{2ll'y - (l+l')x\} = b^2ll' - ab(l+l') + a^2 - 1.$$

Therefore, on re-arrangement,

$$\tfrac{1}{2}y\{-b(l-l')^2\} = b^2ll' - ab(l+l') + a^2 - 1,$$
$$(x-ly)b(l-l') = b^2ll' - ab(l+l') + a^2 - 1.$$

The common right-hand side can be put in the form

$$(a-bl)^2 - 1 + b(a-bl)(l-l')$$

and hence the equations can be written, taking $l - l'$ as the parameter μ,

$$(x-ly) - (a-bl) = A\mu,$$
$$-y = 2A\mu^2 + B\mu,$$

where $A = \{(a-bl)^2 - 1\}/b$ and $B = 2(a-bl)$.

Thus the locus of the point P is the *complete* parabola

$$2b\{x - ly - (a-bl)\}^2 + y\{(a-bl)^2 - 1\} + 2b(a-bl)x = 0.$$

This parabola evidently passes through the points

$$D(0, 0)\ ;\ E\{\tfrac{1}{2}(a+1),\ \tfrac{1}{2}b\}\ ;\ F\{\tfrac{1}{2}(a-1),\ \tfrac{1}{2}b\}$$

and has its axis parallel to $x - ly = 0$. This establishes result (B).

4. Let L and L' be at right angles, so that $ll' = -1$. Then the result of eliminating l and l' from equations (i) and (ii), using this condition, is

$$2b(x^2 + y^2) + (a^2 - b^2 - 1)y - 2abx = 0.$$

This is the equation of the circle passing through the points $(0, 0)$, $\{\tfrac{1}{2}(a+1), \tfrac{1}{2}b\}$, $\{\tfrac{1}{2}(a-1), \tfrac{1}{2}b\}$. Thus if L and L' are at right angles, the point $P(L, L')$ lies on the nine-points circle of the triangle ABC.

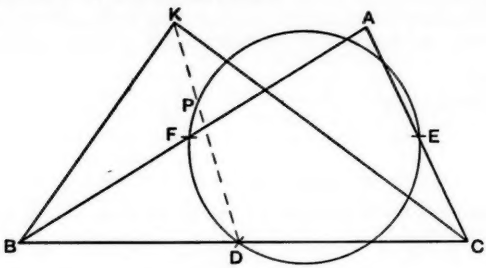

Fig. 2.

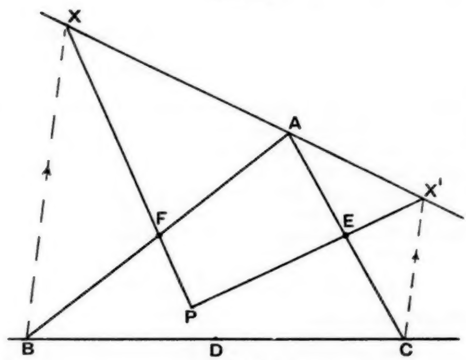

Fig. 3.

Now let P be any point on the nine-points circle of the triangle ABC (Fig. 2). Let K be a point on DP such that $DK = BD = DC$. Then the directions BK and KC are at right angles and DK becomes the other diagonal of the parallelogram on BC as diagonal with sides parallel to the directions BK and CK. Hence $P(L, L')$, for these directions BK and CK, lies on DK and it must also lie on the nine-points circle, by what has just been proved. Hence $P(L, L')$ coincides with P. Thus to every point P of the nine-points circle there correspond real directions L and L' which are at right angles and such that $P(L, L')$ is P. Hence if L and L' are at right angles, $P(L, L')$ traces out the *complete* nine-points circle of the triangle ABC.

This establishes the result (C).

Incidentally we can deduce the following familiar result : (D) If two real parabolas passing through three points are such that their axes are at right angles, then the locus of their fourth point of intersection is the circle passing through the three points, from (B) and (C).

Finally, the results proved here enable us to give : (E) A geometrical construction for the parabola passing through the midpoints of the sides of a triangle and having its axis parallel to a fixed direction, by means of a ruler alone, given the vertices of the triangle, the midpoints of the sides, and lines parallel to the fixed direction through two of the vertices.

Let ABC be the triangle with D, E, F as the midpoints of BC, CA, AB respectively, and let BX, CX' be lines through B and C parallel to the fixed direction. Then if any line through A meets these parallel lines in X and X' respectively and XF, $X'E$ intersect in the point P (Fig. 3), then P is a point of the parabola which passes through the points D, E, F and has its axis parallel to the fixed direction. As the direction XAX' rotates about A, we get all the points of the parabola. The proof is evident.

P. S. R.

1798. " The present *Course*, which I am now engaged in, being the 121st since I began at *Hart-Hall* in *Oxford* in the year 1710. The satisfaction we enjoy by being in any way instrumental to the Improvement of others, is so great, that I can't help boasting—that of eleven or twelve Persons, who perform *Experimental Courses* at this Time in *England*, and other Parts of the World, I have had the Honour of having Eight of them for my *Scholars* whose further Discoveries become an Advantage to myself ; for what would raise Envy in any other Profession, but that of a *Philosopher*, is received as a new Acquisition by all Lovers of *Natural Knowledge*, the Profit being shared in common while the Discoverer has only the Honour of the Invention.—J. T. Desaguliers, *Course of Experimental Philosophy*, Vol. I, 1734, Preface. [Per Dr. M. L. Cartwright.]

1799. There were indeed, about the same time, *Experiments* shown by the late Mr. *Hauksbee*, which were *electrical*, *hydrostatical* and *pneumatical* : But as they were only shewn and explained as so many curious *Phenomena*, and not made Use of as *Mediums* to prove a Series of philosophical Propositions in a mathematical Order, they laid no such Foundations for true Philosophy as Dr. *Keill's Experiments* ; tho' perhaps performed more dexterously and with a finer *Apparatus*.—J. T. Desaguliers, *Course of Experimental Philosophy*, Vol. I, 1734, Preface. [Per Dr. M. L. Cartwright.]

1800. " Well, do you think he looks like a mathematician ? "

" I don't know! How should I know what a mathematician is supposed to look like ? "

" Now you've said something very much to the point! A mathematician doesn't look like anything! Which means, he will always look so generally intelligent that there is no single definite thing behind it at all! With the exception of the Roman Catholic clergy, there is no one these days, absolutely no one, who still looks like what he should look like, for we use our heads even more impersonally than we use our hands. But mathematics is the peak of it all, it has got to the point of knowing as little about itself as human beings—some day when they are living on energy pills instead of meat and bread—are likely to know about meadows and little calves and chickens! "—Robert Musie, *The Man without Qualities*. [Per Cmdr. E. R. Dawson.]

MATHEMATICAL MONSTERS*

By J. L. B. Cooper.

Tonight I shall deal with some examples of mathematical reasoning which have in common only one feature : that of being surprising, or, even, of once having been surprising. We may picture them as monsters, waiting to leap on our loose ideas about mathematics and to tear them to shreds.

I shall not concern ourselves with elementary errors. Every one of you, no doubt, can " prove " that all triangles are isosceles, and that $2=1$ by simple algebra.

The paradoxes and antinomies I shall deal with go deeper : they involve a conflict between logical and intuitive ideas about mathematics. The justification for talking about them is, firstly, I hope, that you will find them entertaining : but there are more serious reasons. In most of our mathematical teaching we are mainly concerned with questions of technique. Logic takes second place : almost all students coming to college know how to differentiate while knowing only hazily what the process means. To develop a more critical attitude needs more than logic : it needs shock tactics, the demonstration that blind following of technique leads to error. And since intuition plays a large part in all creative mathematical thinking, similar shock tactics are necessary throughout all our mathematical life. You cannot persuade most mathematicians to abandon a mode of argument by the statement that it is illogical: actual demonstration that it leads to error is needed, and rightly so, for universal success of a supposedly illogical argument creates in itself a mathematical problem. Trying to do mathematics by logic alone is like finding your way purely by compass across an uncharted region : the antimonies and paradoxes, properly understood, put us in the position of a man keeping to a road by avoiding the rough patches on the side of it.

Perhaps I may continue my zoological studies by classification. In the genus of teratologi, we distinguish three main species, in increasing order of domestication :

(1) Antinomies—(2) Paradoxes—(3) Gegenbeispielen.

The antinomies are the very bad beasts, who bring two arguments both of which seem strong and logical into collision, and really ruin a theory. The paradoxes involve a clash between logic and intuition : and intuition must give way and so solve our difficulty. The gegenbeispielen are the domesticated specimens, doing the bread and butter work of destroying hypotheses which someone might think true, but has no compelling reason for believing.

The teratologi change their class with time : they tend to become more domesticated : to be more concrete, the effect of studying them, and other things in mathematics, broadens the scope of our intuition in mathematics. This intuition is, after all, nothing but a vague unformulated summing up of experience, and widens with experience.

For example, historically one of the most devastating of our monsters was $\sqrt{2}$. The irrationality of $\sqrt{2}$ was discovered by the Pythagoreans ; and it was in sharp conflict with their doctrines about number, their theory that the entire universe was a matter of number, of integers. That the ratio of two lengths could not be a ratio of integers was shattering to them. Even outside their sect, it involved a crisis in Greek mathematics : for the Greeks had in-

* Presidential address to Cardiff Branch of the Mathematical Association, 5th November, 1952.

herited from the Babylonians some methods of algebraic calculation, and notations suitable for expressing rational numbers, but had no means of expressing irrationals. Consequently they could not regard $\sqrt{2}$ as a number, and hence as a quantity with which one could deal algebraically, and if this were true for one ratio of lines, it must be true for all. This was perhaps one reason why the Greeks banished the use of algebra from geometry: and took from Eudoxus their theory of incommensurables, which enabled them to deal with these ratios without reading them as being numbers.

From about the same time come the famous paradoxes of Zeno about motion: the paradox of Achilles and the tortoise, and the others. You will find, if you question your non-mathematical acquaintances, that many of them have the ideas about motion which involve these paradoxes; especially the paradox about the arrow in flight, which cannot be in a definite place at an instant of the time since, if it were, it could not be moving. This paradox depends for its intuitive strength on the notion that one must distinguish different things, in this case different points of space, by naming them; and since names are discrete, points of space are discrete, like stations on a railway line. We escape it by using the real numbers to name the points of our line: but it is worth remarking that this does not answer the question as to the nature of real motion, though it does indicate a possible representation of it which involves no contradiction. Whether actual motion is of the nature of mathematical motion is a different question—and one answered negatively by Bergson's philosophy, and, indeed, by quantum theory.

The most interesting of the more modern paradoxes are the geometrical ones: I propose to discuss some involving our conceptions of plane curves. Intuitively our picture of a continuous curve is that it is something like a chalk or pencil line, but with no thickness: but this will not do for a definition. The locus of a moving point is better: better still is the analytic definition, that it is the locus of points in a plane given parametrically by $(x(t), y(t))$ where $x(t)$ and $y(t)$ are continuous functions of t with t lying between 0 and 1.

The last thing one would expect a curve to do is to fill an area. Yet Peano showed that this can happen. I shall give you a simplified construction, due to Hilbert, of a curve which fills a square.

Let a square be divided into four equal squares numbered 0, 1, 2, 3, anticlockwise, starting at the bottom left hand corner. Now divide each of these squares into four equal squares, and run round the square 0 clockwise, ending opposite a square of 1; pass into the square of 1, and round 1 anticlockwise into 2, and so on. Carry the division further stage by stage, running continuously round the squares, starting always at the bottom left hand corner, running anticlockwise round the first newly divided square after an even number of divisions and clockwise after an odd number: and alternating clockwise and anticlockwise passages round the squares on each run.

The numbers between 0 and 1 can be expressed as decimals in the scale of 4. If t is expressed as such a decimal, assign to t the square in the first subdivision corresponding to its first decimal place, and so on. In this way, each t is assigned a sequence of squares, one inside the other. The sequence of squares has a limit point: let this be the point $(x(t), y(t))$. It is then easy to see that the points assigned to values of t vary continuously with t, and fill the whole square. It should be mentioned that some points in the square, any point which is on the side of any square in any subdivision, are reached more than once: it is, in fact, impossible to have a continuous curve which covers a square so that each point is reached only once.

Among other oddities, this curve contradicts another property we might expect to hold for continuous curves: it used to be supposed that such a curve should have a tangent at every point, or nearly every point. Actually much

more sedate curves, curves which are graphs of continuous functions, do not have this property: the curve $y=f(x)$ where

$$f(x)=\Sigma\frac{1}{m^2}\sin(m!\,\pi x)$$

is an example.

Before I give you a second geometrical paradox, let me define clearly what is meant by the terms " domain " and " boundary ". By a domain we mean a set which is connected, such that any two points in it can be joined by a curve lying in it: and which is open, that is, such that if it contains a point then it contains a circle about the point. The boundary of a domain consists of the set of points which are not in the domain but which are such that any circle about them contains points of the domain.

It is clear that two domains can have the same boundary—the inside and outside of any closed curve is an example. It is very surprising to find, however, that three domains can all have the same boundary: but this possibility is demonstrated by the following example of Brouwer.

Our domains are constructed as follows. Imagine that we have an island, with two lakes inside it, one hot, one cold. The sea and the lakes are domains, that is they are open and do not contain their shores, the boundaries. Now imagine the following engineering project. One day, canals are constructed from the sea and the lakes in such a manner that the canals do not intersect one another, and so that every point of the remaining dry land has all three sorts of water within a distance of less than a mile from it. In the next half day, the canals are extended so that they still do not intersect and every point of dry land is within half a mile of each sort of water: in the next quarter day, the distance is brought down to a quarter of a mile, and so on. At the end of the second day, the remaining dry land consists of points which are on the boundaries of each of the three domains formed of salt water, cold fresh water and hot fresh water respectively.

I do not wish to deal with specifically analytical paradoxes, for these have something of the air of errors of calculation: but the following is neat enough to quote.

$$\begin{aligned}2\log 2&=\frac{2}{1}-\frac{2}{2}+\frac{2}{3}-\frac{2}{4}+\dots\\&=\left(\frac{2}{1}-\frac{2}{2}\right)-\frac{2}{4}+\left(\frac{2}{3}-\frac{2}{6}\right)-\frac{2}{8}+\left(\frac{2}{5}-\frac{2}{10}\right)-\dots\\&=1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\dots\\&=\log 2.\end{aligned}$$

Hence $\qquad 2=1,$

where in the rearrangement terms of order $2n$ go into the nth place and terms of order $2n+1$ go into the $(2n+1)$st place.

More interesting paradoxes occur in the theory of sets. Some of these, concerned with the counting of infinite sets, were noticed by Galileo. We must understand what we mean by saying that two infinite sets have the same number. This cannot, of course, be defined by counting in any ordinary sense. Equality of number of two sets means that we can establish a relation between them by which, to each member of the first set, is correlated one and only one member of the second set. Such a relation is called a one-one mapping. It is a reasonable definition of equality of number since it corresponds to the notion we have for finite sets: we count a finite set by a one-one mapping of its members onto the set of integers 1, 2, ... n, where n is the number of the set: and, without counting, we can say that the number of seats in a full cinema is

equal to the number of the audience if every seat is occupied by one person and no one is standing, because there is then a one-one relationship between the seats and the audience, the relation assigning to each seat its occupant.

The paradoxes noticed by Galileo are that infinite sets may have the same number even though one of them contains the other. For instance, the points of the interval (0, 1) are mapped onto those of (0, 2) by the one-one mapping $x \to 2x$. Again, the number of even numbers is the same as the number of all numbers. The number of squares, or of primes, is also the same as the number of all numbers. The rational numbers, have the same number : for they can be put in order by assigning the numbers p/q to sets in each of which the number $(p+q)$ is the same, and ordering those in each set by increasing magnitude of q : then we get the ordering

$$\frac{0}{1}, \frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{3}{1}, \frac{1}{4}, \frac{2}{3}, \frac{3}{2}, \frac{4}{1}, \dots$$

from which we have dropped rationals which are repetitions of ones earlier in the series. This list includes every rational, and if the mth is assigned to the mth integer we get a one-one mapping of the integers on the positive rational numbers.

This might lead one to think that all infinite sets have the same number : but this is not the case. A set which has the same number as the integers is called countable, and one can show that the decimals are not countable. For suppose they were countable : then we could write down the decimals from 0 to 1 in order, assigning each to an integer n. One can show that no such assignment of a decimal to an integer can exhaust all the decimals ; if we were given such an assignment, we could form a new decimal by the following process. If the nth decimal in our ordering has a_n in its nth decimal place, make the nth decimal place of our new decimal a_n+1 if $a_n<9$, and a_n-1 if a_n is 9. The new decimal differs from every decimal of the original ordering : for its rth place is different from the rth place of the rth decimal of that set.

The real numbers are thus not countable : in our sense, there are far more of them than there are rationals.

The relative insignificance of the set of rationals comes out more clearly if we consider the problem of measuring of sets, and can be illustrated without going into the details of measure theory.

Suppose that we enclose every rational number p/q between 0 and 1 in an interval $\left(\frac{p}{q}-\frac{1}{q\,.\,2^{k+q}}, \frac{p}{q}+\frac{1}{q\,.\,2^{k+q}}\right)$ where $k>2$. There are q numbers of the form p/q for a given q, so that intervals for these numbers have a total length of not more than 2^{1-k-q}; and the total length covered by all these intervals, not allowing for overlapping, is $2^{1-k}\,(1+\frac{1}{2}+\frac{1}{4}+\dots)=2^{2-k}$.

By taking k large we can make this total length as small as we please : and so we have the entire set of rationals enclosed in a set of intervals of total length as small as we like. It is interesting to consider the remaining set of numbers. This includes all the interval (0, 1) save for a set of intervals of arbitrarily small total length, say all save $1/2^6$ if $k \geqslant 8$. Yet this residual set does not contain a single interval : for it contains no rational numbers, and between any two real numbers lies a rational number. Thus the residual set is made up of isolated points, in spite of which it forms, in measure, the bulk of the interval.

While no-one is likely to find this paradox difficult to accept, measure theory has at least one monster so fierce that eminent mathematicians dispute its classification. The paradox is connected with the problem of measure. One form of this problem is : can one assign a measure to every set of the line which has the following properties :

(1) if one set can be displaced, by a rigid body displacement, to coincide with a second, the two sets have the same measure ;

(2) if a set is the sum of two disjoint sets, its measure is the sum of their measures.

Now it can be proved, by highly sophisticated methods, that the problem is soluble for the sets of the line. It can also be shown to be soluble for sets in the plane. For sets in space, however, the problem is insoluble. This was first proved by Hausdorff : and his argument is based on the construction of a division of the surface of a sphere into three disjoint sets, A_0, A_1, A_2 which together contain all the sphere save for a countable set, and are such that A_0 can be rotated to coincide with A_1, or with A_2, and can also be rotated to coincide with $A_1 \cup A_2$, the union of A_1 and A_2.

To construct these sets we proceed as follows. Let C be the centre and O a point on a unit sphere and let ρ denote the operation of rotation through 180° about CO, so that $\rho^2=1$. Let P be another point of the sphere such that the angle OCP is a transcendental multiple of π, say π/e ; and let σ denote the operation of rotation through 120° about CP, so that $\sigma^3=1$.

We now consider the set of all products of these rotations, i.e. all rotations of the form $\sigma^k\rho\sigma^{n_1}\rho\sigma^{n_2}\rho \ldots \rho\sigma^{n_\nu}\rho^l$, where the n's are all 1 or 2, $k=0$, 1 or 2, $l=0$ or 1, and we divide the set of rotations into three sets R_0, R_1, R_2 corresponding to the three values of k. The point of choosing OCP to be a transcendental multiple of π is that then no two of these rotations coincide. If we exclude from the sphere the set of points, say Z, which are kept invariant by any one rotation, then Z is only a countable set, and if we take any other point of the sphere it is taken into different points by any two different rotations of the set.

Choose such a point, say Q. Let B_0, B_1, B_2 denote the sets into which Q is taken by the rotations of R_0, R_1, R_2 and let $B=RQ$ be the sum of these sets.

$B_0=R_0\ Q$ is the set of all points of form $\rho\sigma^{n_1}\rho \ldots \sigma^{n_\nu}\rho^l Q$,
$B_1=R_1\ Q$ is the set of all points of form $\sigma\rho\sigma^{n_1}\rho \ldots \sigma^{n_\nu}\rho^l Q$,
$B_2=R_2\ Q$ is the set of all points of form $\sigma^2\rho\sigma^{n_1}\rho \ldots \sigma^{n_\nu}\rho^l Q$.

Then it is clear that the application of the rotation σ to B_0 gives a displacement of this set onto B_1, σ^2 gives a displacement of it onto B_2, while ρ rotates it so that it coincides with $B_1 \cup B_2$.

Here we have the essence of the paradox. To get the answer to the problem of measure theory, we have to spread out the sets to which it applies. To do this, we consider the sphere as divided up into sets of the form of B, sets into which any one point not in Z is taken by the rotations of R. These sets are non-overlapping, and together occupy all the sphere except for Z. Choose one point Q' from each set, and from each Q' form the corresponding sets B_0', B_1', B_2'. Let A_0 be the union of all the sets B_0, A_1 of all the sets B_1, A_2 of all the sets B_2. Then A_1, A_2, A_3 fill all the sphere save for the countable set Z ; and

$$\sigma A_0=A_1,\ \sigma^2 A_0=A_2,\ \rho A_0=A_1 \cup A_2.$$

This paradox has engaged the attention of the great French mathematician Borel. He regards the result as impossible, and argues that a fallacy is involved in the assumption that we can choose a point Q' from each of the sets B ; this selection would be justified by an appeal to the Axiom of Choice, and Borel considers that this paradox, which he would regard as an antimony, condemns the axiom of choice. This point of view is not shared by others. Indeed to my mind, and this was the motive for the way I set out the paradox, the essential paradox is already there in the result for the sets B_0, which does not involve the axiom : application of the axiom leads to a result which is more instructive, but not more paradoxical.

Even odder results have been obtained on these lines. Thus von Neumann states that one can show that a sphere can be divided into 9 subsets, four of which can be displaced to fit together into a sphere identical with the original, and the other 5 to fit together into a second sphere identical with the original.

The axiom of choice brings us to the most fundamental parts of mathematics, the theory of sets; and it is in this that some of the most ferocious monsters have their hunting grounds: almost untameable antinomies, whose dangerous power is that they seem to attack a type of thinking almost universal in mathematics, our notions of class and set themselves.

In a way these antinomies are old: they can be traced back to a gentleman who has gone down in history under the title of Epimenides the liar, not because he was untruthful, but for making what may well have been the correct statement that "All Cretans are liars"; the point being that he was himself a Cretan. King David is credited with the statement that all men are liars. The vicious circle involved in these statements is easily broken: but it is not so easy to resolve the difficulties involved if I say "The sentence which I am speaking now is false and its opposite is true."

Antinomies of this type attracted attention at the end of last century and the beginning of this. It was hoped at that time that one could place mathematics on a permanently secure foundation by basing it on the theory of classes and on nothing else. This programme was begun by certain continental mathematicians, Peano and Frege, and its best known exposition is that of Russell and Whitehead, who in their *Principia Mathematica* aimed to show that mathematics is a branch of logic. Unfortunately, before that work was published the attempt it represented had run into almost insuperable difficulties through these antinomies, and *Principia* was able to give only a partial solution.

Two typical antinomies are due to Richard and Russell.

Richard's antinomy is this. Let n be the smallest number which cannot be defined in the English language in less than 50 words. Then I have used less than 50 words in defining n, so a contradiction is involved.

Secondly, consider Russell's antinomy. Some classes are not members of themselves, for instance the class of horses is a class, not a horse, so is not a member of itself. On the other hand, the class of all classes is a class and so is a member of itself. Now consider the class S of all classes which are not members of themselves. If S is a member of itself, it is not a member of itself, and if S is not a member of itself, it is a member of itself.

It is extremely awkward to find that notions which seem perfectly natural should involve contradictions, and a great deal of thought has been given to formulating systems for the foundation of mathematics which shall be free of these antinomies. Some mathematicians have had the hope of ensuring in some way that mathematics can be founded with a guarantee of freedom from all antinomies: but it is practically certain that this cannot be done without restricting the types of reasoning allowed in mathematics to an extent which most mathematicians would consider intolerable. The point is that we have a choice between keeping our mathematical reasoning always within a tightly drawn fence, safe from the outside monsters: or, while taking reasonable precautions, allowing ourselves larger pastures in which we run the risk of being devoured by some mathematical monster. The latter is surely the way most mathematicians would prefer: certainly all those who regard their subject as a science, for no science is free from the risk of error, and the risk of error in mathematics is small enough in all conscience as long as we keep to the generally accepted methods, even if these do not have a certificate of freedom from contradiction from the mathematical logicians.

The examples of paradoxes I have given are all from Pure Mathematics.

This is not because Mathematical Physics and Applied Mathematics cannot be a source of paradoxes, but because theirs are usually of a different nature. For instance, many things in relativity theory, and even more in quantum theory, are very paradoxical : but the paradoxes arise from a conflict between the picture of the world developed by Newton and his successors on the basis of experience of bodies of ordinary size and the modern picture of cosmological or atomic phenomena, whereas the paradoxes I have discussed are of a logical nature. Such paradoxes can also occur in Mathematical Physics, and can be instructive. For instance, it is paradoxical in the first sense that one cannot measure both the position and momentum of an electron simultaneously : this conflicts with our picture of an electron as a particle. A stranger paradox was found by Einstein, Rosen and Pololsky : they showed that under certain circumstances, when two electrons are brought together and then separated, one should be able, by doing measurements first on the one and then on the other, to measure both the position and the momentum of the second. This is a paradox of a logical nature, and affects our ideas on the foundation of quantum theory ; but I believe that it can be shown that this paradox rests on a direct error in mathematics, and that it is interesting as showing that, in these branches of Physics, the use of nonrigorous arguments common in other branches of mathematics is breaking down, because the support from intuition which the physicist has in more commonsense subjects is being removed.

Perhaps I may close with a few examples from elementary dynamics. The first is particularly instructive for engineers. Consider the rocket, driven by discharging gas backward at a rate of m units of mass per second with a relative velocity v. The thrust on the rocket is then mv, and the work put into the discharge is $\frac{1}{2}mv^2$. If the rocket is travelling at a steady speed V, the rate of doing work against outside resistance is mvV. Hence the efficiency of process is

$$\frac{mvV}{\frac{1}{2}mv^2}=\frac{2V}{v}$$

and if $v=V$, the efficiency is 2. This is a thing which engineers in particular find disconcerting : it was the subject of a long correspondence in *Flight*. Efficiencies greater than 1 occur in similar connections : for example, if the cooling air of an aircraft is given a backward thrust by a fan, the efficiency of the process, measured as the ratio of the work done on the aeroplane to the work done by the fan, can be greater than 1.

There are some more puzzling paradoxes connected with the laws of friction, which were first pointed out by Painlevé. The following is one of the simplest. Let two unit masses m and m' run along two parallel lines and let them be joined by a rigid rod inclined at an angle α to the lines. Let m be pulled along its line by a force X, and let m' have a coefficient of friction μ with its line. Then the equations of motion are

$$\ddot{x}=X-T\cos\alpha$$
$$\ddot{x}=T\cos\alpha-F,$$
$$F=\epsilon\mu T\sin\alpha$$

where x measures the positions of m and m', T is the tension in the rod, F the friction on m' measured in the direction opposite to the positive direction of x and X, and $\epsilon=\pm 1$.

Then $$T=X/(2\cos\alpha-\epsilon\mu\sin\alpha).$$

Let us suppose that the rigid rod is at a large angle to the lines, so that $\tan\alpha>2/\mu$.

In this case, the sign of T is the same as the sign of ϵ, and hence ϵT is positive. If, then $\dot{x} < 0$, we have incompatibility with the condition $\epsilon T\dot{x} < 0$, while if $\dot{x} > 0$, $\epsilon T\dot{x}$ is positive whichever sign ϵ and therefore T have, so that two motions are compatible with the conditions.

Moreover, in the motion with ϵ negative, the acceleration is negative, in spite of the positive force pulling the system.

Similar paradoxes occur in very many problems concerning friction ; for example, if one considers a solid cylinder standing on a rough inclined plane, and moving down it, then if the angle of inclination is so steep that the cylinder should topple, and the coefficient of friction is large enough, in fact if

$$\tan\alpha > \frac{r}{l}, \quad \mu > \frac{k^2}{rl}$$

where r is the radius of the cylinder, l its height, and k the radius of gyration about a line in the base through the axis, we get a similar incompatibility. These things led Painlevé to maintain that Coulomb's laws of friction are not merely incorrect in practice, which is likely enough, indeed certain, but that they are logically incompatible with the laws of mechanics. This last view is, I think, untenable : what must happen in these cases of incompatibility is that the friction must act as an instantaneous brake, bringing the motion to rest. This is surprising enough : and these paradoxes do demonstrate an actual logical incompatibility between the laws of friction and of mechanics, the assumptions about rigid bodies, and our reasonable assumptions about continuity of velocity.*

J. L. B. C.

* For a recent discussion see G. Hamel, *Theoretische Mechanik*, (Springer 1949), pp. 543–549, 629–636.

1801. It is in any case quite obvious to most people nowadays that mathematics has entered like a daemon into all aspects of our life. Perhaps not all of these people believe in that stuff about the Devil to whom one can sell one's soul ; but all those who have to know something about the soul, because they draw a good income out of it as clergy, historians, or artists, bear witness to the fact that it has been ruined by mathematics and that in mathematics is the source of a wicked intellect that, while making man the lord of the earth, also makes him the slave of the machine. The inner drought, the monstrous acuity in matters of detail and indifference as regards the whole, man's immense loneliness in a desert of detail, his restlessness, malice, incomparable callousness, his greed for money, his coldness and violence, which are characteristic of our time, are, according to such surveys, simply and solely the result of the losses that logical and accurate thinking has inflicted on the soul! And so . . . there were people who were prophesying the collapse of European civilization on the grounds that there was no longer any faith, any love, any simplicity or any goodness left in mankind ; and it is significant that these people were all bad mathematicians at school. This only went to convince them, later on, that mathematics, the mother of the exact natural sciences, was also the grandmother of engineering, the arch-mother of that spirit from which in the end, poison-gas and fighter-aircraft have been born.—Robert Musie, *The Man without Qualities*. [Per Cmdr. E. R. Dawson.]

SOME REMARKS ON THE GENERAL POWER AND EXPONENTIAL FUNCTIONS

By A. W. Gillies.

The mathematics teacher has from time to time to make a decision as to how much he should say to a class about a topic which is too difficult to discuss with full mathematical rigour. This situation may arise with students who are too young or too immature in their mathematical development to appreciate a rigorous discussion, or with older students who are interested in mathematics purely as a tool which they require in the study of other subjects such as engineering and have not time to spare even if they would be interested in a rigorous development of the particular topic.

Such decisions have to be made in a manner depending on the particular circumstances of the course and on the type of student. In the opinion of the present writer the student should be given as logical a development as he is capable of appreciating or as time permits, and when assumptions are made they should be clearly stated. Whatever is said should be convincing to the student at his present level, and at the same time should prepare the way for a more rigorous treatment at a later stage should the student proceed so far with his mathematical studies. The one thing which is inexcusable is to pretend that the difficulty does not exist.

The writer has particularly in mind the question of irrational numbers which arises at the intermediate or G.C.E. advanced level stage when index laws and surds have to be discussed and following on these the differential coefficient of x^n has to be established.

While pondering on these matters recently the writer referred to several calculus texts of intermediate standard to see what they had to say with regard to the differential coefficient of x^n. He was rather surprised to find that in general they had nothing whatever to say. The differential coefficient was usually obtained for n rational and the possibility of an irrational index was not even mentioned.

More advanced calculus texts evade the difficulty by dealing with the exponential and logarithm functions first. Lamb (*Infinitesimal Calculus*) obtains the exponential function by a series solution of $\frac{dy}{dx}=y$ followed by a rather elaborate demonstration that the series represents a differentiable function. It is then shown that the function coincides with e^x as defined in elementary algebra for x rational, and this function then defines e^x for x irrational, it being shown that the index laws are satisfied. Hardy (*Pure Mathematics*) defines $\log x$ first as $\int_1^x \frac{dt}{t}$ from which e^x is obtained as the inverse function. The general power is then defined by $a^x=e^{x\log a}$, the index laws are shown to be satisfied and the differential coefficient $\frac{d}{da}a^x=xa^{x-1}$ is derived. There is a note to the effect that the results of this section enable the restriction to rational indices to be removed from many earlier results.

De la Vallée Poussin defines the general power a^n for n irrational as $\lim\limits_{r\to\infty} a^{n_r}$, n_r a sequence of rational numbers with limit n, but proceeds from this to the exponential limit from which the differential coefficients of e^x and $\log x$ are obtained. Finally x^n is written $e^{n\log x}$ in order to obtain its differential coefficient for all values of n.

Returning to the intermediate student, he must certainly encounter

irrational numbers in a fairly formal manner in connection with quadratic surds and it seems quite unsatisfactory to ignore them completely either in dealing with a^n or with the differential coefficient of x^n, for non-integral values of n. It must be remembered that the future students of the mathematical degree courses in the universities will pass through this stage as well as many others who will carry their mathematical studies to an advanced level even if from a more utilitarian point of view.

At a more advanced stage the procedures of Hardy and de la Vallée Poussin have their attractions, but it is still true that the student first meets the exponential and logarithm via the index laws and the first differential coefficient he meets is that of x^n. To change round suddenly and define $\log x$ by an integral, and to replace the, to him, simpler function x^n by the more complicated $e^{n \log x}$ must seem an unnecessarily indirect approach, which is more than ever puzzling when no attempt is made to explain the advantage of this indirect procedure or to show the difficulty of a direct approach.

The only text in which the writer has found a discussion of this question is that of Courant (*Differential and Integral Calculus*, Vol. I). Here the general power a^α (α irrational) is introduced as $\lim_{m\to\infty} a^{r_m}$ with r_m a sequence of rational numbers having the limit α. The integral of the general power is obtained from $\int_a^b x^\alpha\,dx = \lim_{m\to\infty}\int_a^b x^{r_m}\,dx$ and from this the differential coefficient is inferred, it being shown that the difference $|x^\alpha - x^{r_m}| < \epsilon$ for $m > m_0$ and all x in (a, b) i.e. essentially that x^{r_m} converges uniformly to x^α in (a, b) although the term "uniform convergence" is not used at that stage. Later it is shown geometrically that the difference between two functions may be uniformly small in an interval without the same being true of their derivatives and consequently the differential coefficient of x^α cannot be obtained in the same way as was done for the integral by a passage to the limit from x^{r_m}. However no attempt is made to inquire whether this passage to the limit could be justified in the particular case.

At a later stage still, Courant reintroduces the logarithm by the integra definition and develops the exponential and power functions from this. He concludes the discussion by pointing out that this new approach largely avoids the troublesome limiting processes and discussions of continuity required by the " elementary " approach, because the definition of the logarithm as the integral of a positive function automatically ensures a continuous differentiable function having a continuous differentiable inverse function.

Thus Courant's development is quite the most illuminating and instructive. His method of starting with the integral, a number of elementary integrals being obtained directly from the summation definition of the definite integral before he turns to the derivative, is the opposite of traditional practice in this country, and his approach is too mature in outlook for the beginner. The remainder of this article outlines a scheme suitable for classes of intermediate standard which is thought to meet most points raised in the foregoing.

Firstly on the basis of the index laws the definition of the power a^r is extended to rational indices in the usual way. It is then shown that a^r is monotonic, increasing for $a > 1$, decreasing for $a < 1$, and continuous i.e. the difference $|a^{r_1} - a^{r_2}|$ may be made arbitrarily small if $|r_1 - r_2|$ is sufficiently small. Now define a^α by (taking $a > 1$)

$$a^{r'} < a^\alpha < a^{r''}$$

where r' and r'' are *any* rational numbers less than α and greater than α respectively. The student will easily appreciate that this defines a unique

number which can be approximated to any desired degree of accuracy by taking r' and r'' sufficiently near together, thus $a^{\sqrt{2}}$ lies between $a^{1 \cdot 414}$ and $a^{1 \cdot 415}$ and so on. Furthermore this means geometrically simply drawing the continuous curve through the points already obtained for rational indices. The exponential graph is thus obtained. The more convenient formulation $a^\alpha = \lim_{m \to \infty} a^{r_m}$ with r_m any sequence of rational indices having limit a, e.g. the successive decimal approximations, is then easily appreciated. The logarithm is next obtained as the inverse function, namely the index which expresses the number as a power of the base. The fundamental laws for the logarithm then appear as the index laws over again.

Next, as a preliminary to the differentiation of x^n, $\lim_{x \to a} \dfrac{x^n - a^n}{x - a}$ is considered. It is shown in the usual way that the limit is na^{n-1} for n a positive integer, zero, and positive rational number in succession. For $n = \alpha$ a positive irrational number, suppose first that $x > a \geqslant 1$. For any positive n write

$$x^n - a^n = a^n \left[\left(\frac{x}{a} \right)^n - 1 \right],$$

and it is clear that the second factor on the right increases, and the first factor does not decrease, with increasing n, and so we have

$$x^{r'} - a^{r'} < x^\alpha - a^\alpha < x^{r''} - a^{r''}.$$

If $x < a$ the inequalities are reversed and in either case division by $x - a$ gives

$$\frac{x^{r'} - a^{r'}}{x - a} < \frac{x^\alpha - a^\alpha}{x - a} < \frac{x^{r''} - a^{r''}}{x - a}$$

and letting $x \to a$ gives

$$r' a^{r'-1} \leqslant \lim_{x \to a} \frac{x^\alpha - a^\alpha}{x - a} \leqslant r'' a^{r''-1}$$

for any $r' < \alpha$ and $r'' > \alpha$. Thus the middle member can have no other value than $\alpha a^{\alpha - 1}$.

For $a < 1$ we now write $a = \dfrac{1}{b}$ and $x = \dfrac{1}{y}$ with $b, y > 1$ and the result may then be established using the case already proved. Finally n any negative number is dealt with in the usual way.

The $\lim_{x \to a} \dfrac{x^n - a^n}{x - a}$ is thus established without serious difficulty for all n, and the differential coefficient of x^n can now be obtained without restriction.

For the exponential function choose a temporary standard base b (e.g. 10). We have easily

$$\frac{db^x}{dx} = kb^x$$

where $$k = \lim_{h \to 0} \frac{b^h - 1}{h} = \left[\frac{db^x}{dx} \right]_{x=0}.$$

For any other base we have

$$y = a^x = b^{x \log_b a}$$

and $$\frac{dy}{dx} = k \log_b a \, . \, a^x.$$

Thus for any exponential the differential coefficient is proportional to the function. Since $\log_b a$ is monotonic increasing (if $b>1$) there will be one and only one value $a=e$ for which the constant of proportionality is 1. The number e is then given by

$$k \log_b e = 1$$

or

$$e = b^{\frac{1}{k}}$$

The number e may be determined approximately from a graph of $y=b^x$ by drawing the tangent at the point where it crosses the y-axis. If this tangent meets $y=2$ at Q, then Q has abscissa $\frac{1}{k}$ and the ordinate to the curve through Q is of length e.

Alternatively writing $b^h - 1 = u = \frac{1}{n}$ gives

$$\frac{1}{k} = \lim_{u\to 0} \log_b (1+u)^{\frac{1}{u}}$$

and so

$$e = \lim_{u\to 0} (1+u)^{\frac{1}{u}}$$

$$= \lim_{u\to\infty} \left(1+\frac{1}{n}\right)^{\frac{1}{n}}$$

which expresses e in a form independent of b.

The logarithm and $\int \frac{dx}{x}$ then follow quite easily.

This development avoids many of the difficulties. The only assumption is the existence of the limit defining k which, since it is the gradient at the point $x=0$ of the curve may reasonably be assumed to exist on grounds of geometrical intuition, but it is always possible to fill this gap by a formal discussion of the limit either immediately or at a later stage. Throughout the development the details may be filled in rigidly and formally or relatively loosely according to the nature of the class.

The integral definition of the logarithm is then left to a later stage for students who are specialising in mathematics, or it may be avoided altogether for other types of student for whom it does not have much appeal.

A. W. G.

1802. Another significant feature introduced into Chandigarh is the Modulor. The Modulor, a scale of mathematical dimensions taken from the human body and invented in 1942, has been applied in the construction of the Boulevard Michelet building at Marseilles, and is spreading more and more widely over the world. Here, at just the right moment for town-planning, is a harmonic scale based on the relevance of human beings to every architectural site. By the miracle of numbers, the reduction of dimensions to a common measure, or to a measure of the same harmonic type, becomes the indispensable intermediary step for prefabrication and mass production. The words " mass production " will no longer be a byword for monotony and boredom. Thanks to the Modulor, limitless diversity and mathematical harmony are being reintroduced into the surroundings of men's lives.—*The Observer*, " Le Corbusier ", May 10, 1953. [Per Mr. H. V. Lowry.]

MATHEMATICAL NOTES

2452. *Bifid Operators.*

1. Introduction.

The uses and properties of the operator p, used to represent the operation of differentiating, are well known. In what follows bifid operators of a similar kind are devised for dealing with functions of the type used to denote amplitude-modulated alternating quantities, *i.e.* functions which can be expressed as the product of two periodic functions of time, the two periods being unequal. These operators however appear to be of use with any function which can be expressed as a product of two functions of the same variable.

2. Bifid operators.

The operators p_m and p_c will be defined by the equations

$$p_m(m \,.\, c)=\frac{dm}{dt} \,.\, c,$$

$$p_c(m \,.\, c)=m \,.\, \frac{dc}{dt},$$

where m and c are both functions of t. The operator p_m represents a differentiation of the function $(m \,.\, c)$ in which c is regarded as a constant, and similarly p_c differentiates c but regards m as a constant.

3. Some properties of the operators.

3·1. The derivative of a product is given by

$$\frac{d(m \,.\, c)}{dt}=\frac{dm}{dt} \,.\, c+m \,.\, \frac{dc}{dt}$$

or, in operational form,

$$p(m \,.\, c)=(p_m+p_c)(m \,.\, c)$$

i.e.

$$p=(p_m+p_c).$$

3·2. Using the operation $1/p_m$ to denote the inverse of p_m in the usual way

$$\frac{1}{p_m}(m \,.\, c)=c\int m\,dt, \quad \text{and} \quad \frac{1}{p_c}(m \,.\, c)=m\int c\,dt.$$

This is verified by operating on both sides of the two equations by p_m and p_c respectively.

3·3. The integral of a product is

$$\int (m \,.\, c)\,dt=m\int c\,dt-\int\left(\left(\frac{dm}{dt}\right) . \int c\,dt\right)dt$$

or, in operational form,

$$\frac{1}{p}(m \,.\, c)=\left(\frac{1}{p_c}-\frac{1}{p}\frac{p_m}{p_c}\right)(m \,.\, c)\ ;$$

operating by p_c

$$\left(\frac{p_c}{p}+\frac{p_m}{p}\right)(m \,.\, c)=(m \,.\, c)$$

i.e.

$$\frac{1}{p}=\frac{1}{p_m+p_c}.$$

3·4. In section 3·3 above, no justification for changing the order of the operators is given. In general, however, the operators obey all the rules of

algebra, *i.e.* functions of p, p_m, and p_c may be added or multiplied together in any order, and the binomial theorem may be used to expand functions of the operators. No general proof of this is given, but it is easy to verify in particular cases.

3·5. Differentiating n times,

$$p^n(m\,.\,c)=(p_m+p_c)^n(m\,.\,c)\,;$$

expanding by the binomial theorem

$$p^n(m\,.\,c)=\left(p_m^n+np_m^{n-1}\,p_c+\frac{n(n-1)}{2!}\,p_m^{n-2}\,p_c^2+\ldots p_c^n\right)(m\,.\,c).$$

This corresponds to Leibnitz's Theorem.

3·6. Similar expressions to that given in section 3·5 may be obtained when n is negative. For example

$$\frac{1}{p}=\frac{1}{p_m+p_c}=\frac{1}{p_m}\left(1-\left(\frac{p_c}{p_m}\right)+\left(\frac{p_c}{p_m}\right)^2-\left(\frac{p_c}{p_m}\right)^3+\ldots.\right)$$

This result may be used to write down the integral of expressions such as $(t^3 \sin t)$ immediately. Taking m as $\sin t$, and c as t^3, we have

$$\int(t^3\sin t)dt=-t^3\cos t+3t^2\sin t+6t\cos t-6\sin t+K$$

The procedure is as follows

(*a*) First operate on $(m\,.\,c)$ by $\dfrac{1}{p_m}$ giving $(-t^3\cos t)$,

(*b*) operate on the result by $\left(-\dfrac{p_c}{p_m}\right)$ giving $(+3t^2\sin t)$,

(*c*) operate again by $\left(-\dfrac{p_c}{p_m}\right)$ giving $(+6t\cos t)$

(*d*) and again giving $(-6\sin t)$,

(*e*) and proceed in this manner until the result is zero.

The integral is given by the sum of the individual results (*a*), (*b*), (*c*), (*d*), plus a constant of integration K.

The order of the operations is interchangeable ; the same result is obtained by first operating with $\left(1-\dfrac{p_c}{p_m}+\ldots\right)$, and then operating on the result with $\dfrac{1}{p_m}$.

In general the nth integral of a product is

$$(p_m+p_c)^{-n}=\left(\frac{1}{p_m}\right)^n\left(1-n\left(\frac{p_c}{p_m}\right)+\frac{n(n+1)}{2!}\left(\frac{p_c}{p_m}\right)^2-\ldots\ldots\right)$$

e.g. the nth integral of $(t^3\sin t)$ is

$$\left(\frac{1}{p}\right)^n(t^3\sin t)=t^3\sin\left(t-\frac{n\pi}{2}\right)+3nt^2\cos\left(t-\frac{n\pi}{2}\right)-\frac{6n(n+1)}{2!}\,t\sin\left(t-\frac{n\pi}{2}\right)$$
$$-\frac{6n(n+1)(n+2)}{3!}\cos\left(t-\frac{n\pi}{2}\right)+\left(\frac{K_1t^{n-1}}{(n-1)!}+\frac{K_2t^{n-2}}{(n-2)!}+\ldots K_n\right).$$

where K_1 K_2, etc. are constants of integration.

4. *Application to Amplitude-modulated alternating quantities.*

We suggest here briefly two examples of the use of bifid operators.

Example 1. Find the current which flows in a pure inductance L when a voltage $v=V\cos\omega_m t\,.\cos\omega_c t$ is applied to it.

The current
$$i=\frac{1}{L}\int v\,dt=\frac{v}{pL}.$$
$$\frac{1}{p}=\frac{1}{p_m+p_c}=\frac{p_m-p_c}{p_m^2-p_c^2}$$

Thus,
$$i=-\frac{p_m-p_c}{\omega_m^2-\omega_c^2}\frac{v}{L}\quad\text{since } p^2=-\omega^2 \text{ when operating on } \cos\omega t$$

and so
$$i=\frac{V}{L(\omega_m^2-\omega_c^2)}(\omega_m\sin\omega_m t\,.\cos\omega_c t-\omega_c\cos\omega_m t\,.\sin\omega_c t).$$

The case of a non-pure inductance (*i.e.* having resistance as well as self-inductance) is easily dealt with, but the method would be more akin to that used in the following problem.

Example 2. To find the output of a four-terminal network having a transfer ratio (or " gain ") $G=1/(1+pT)$, when the input consists of an amplitude-modulated voltage $v=m\cos\omega_c t$ (m is any function of time).
$$G=\frac{1}{1+pT}=\frac{1}{1+p_mT+p_cT}=\frac{1+p_mT-p_cT}{(1+p_mT)^2-p_c^2T},$$
but $p_c^2=-\omega_c^2$; hence
$$G=\frac{(1+p_mT)-p_cT}{(1+\omega_c^2T^2)+2p_mT+p_m^2T^2}.$$

The output will be
$$Gv=\left[\frac{(1+p_mT)m}{(1+\omega_c^2T)+2p_mT+p_m^2T^2}\right].\cos\omega_c t+\left[\frac{\omega_cT\,.\,m}{(1+\omega_c^2T)+2p_mT+p_m^2T^2}\right]\sin\omega_c t.$$

The quantities in the square brackets are functions of p_m and can be evaluated by Laplace Transforms for any given modulating voltage m.

The expression for the output derived here represents the steady state output. A complete solution must contain a transient expression (complementary function) $Ke^{-t/T}$, where K is the constant of integration.

C. J. N. Candy.

2453. *Observations on the nature of demonstrative evidence* (*Thomas Beddoes*).

" But in elementary school work we are faced with a different problem, *viz.* the first recognition of geometrical facts and the development of the power of geometrical perception. Until this has been to some extent acquired attempts to reason on the facts are constantly hampered. The paradox may almost be ventured that in many cases until a fact is obvious, argument will be of little avail." The words of the *Second Report on the Teaching of Geometry in Schools* will be familiar to most of us, and we consider the " attempts to keep a course in geometry free from arguments that even the commonsense logic which is appropriate to the schoolroom must recognise as defective ", as a development in the teaching of the subject which has occurred during the last fifty years. It is, therefore, interesting to read the forthright expression of the same

opinion by Thomas Beddoes, writing in 1792. "The more I consider the subject, the more I am inclined, . . . to believe not only in the possibility, but the utility of rendering the elements of geometry palpable. If they be taught at an early age—a plan in which I think I see many advantages—models would make the study infinitely more engaging : From the mere slate and pencil most beginners experience a repulsive sensation. But if a child had something to handle and to place in various postures, he might learn many properties of geometrical figures without any constraint upon his inclinations. He would have no difficulty in transferring the properties of palpable to merely visible figures, nor in generalising the inferences. . . . We should have laid a good foundation for the invaluable habit of accurate observation in general ; and towards future progress in mathematics, we should have warded off the first disagreeable impression of the aspect of the science, which is so very apt to strike a damp to the heart of the beginner.

"I need not explain to you the advantage of trying to engage Fancy on our side, by all the allurements we can offer to her. It is she that smooths every path and strews it with flowers. We all, men and boys, follow with alacrity wherever she leads ; neither the mind, nor the body, grudge any labour ; and it is the enthusiasm she inspires, that has worked so many miracles in art and science. By some strange fatality, however, she is neglected, if not affronted in almost all the stages of education ; and the first step in almost every species of instruction is, to present knowledge to the student's imagination, in conjunction with some melancholy and hateful accompaniment ; which sort of management, I conceive to have much the same kindly influence upon this faculty as an unseasonable frost upon the tender petals of an expanding blossom.

"The mode of initiation in geometry which I propose, could not, unless I very much deceive myself, fail to render the impressions of sense more agreeable by rendering them more distinct. The rigorously scientific method, as it is supposed to be, seems, on the contrary, to aim only at rendering them as obscure as possible : an intention, I confess, perfectly in union with the other parts of the established process of school and college stupefaction.

"Whether you will allow that this important point is likely to be in this manner attained, I am not sure. But you will agree with me in thinking, that it is high time to discard Euclid's Elements."

Thomas Beddoes, the father of the poet, was born in 1760 and died in 1808. He took his M.D. at Oxford and was in 1788 Reader in Chemistry. His book, *Observations on the nature of demonstrative evidence with an Explanation of certain Difficulties occuring in the elements of Geometry and reflections of Language*, from which this quotation is taken, was dedicated to Davies Giddy who was President of the Royal Society from 1827 to 1830. In many ways this book, which was published 120 years before the *First Report*, anticipates recommendations which it has again been necessary to make in the *Second Report* of 1937. On page 19 of this latter report we read that the remedy for the boy who asserts "that the diagonal of a parallelogram necessarily bisects the angles" is "not argument or proof but demand for the figure he has in mind, and reconsideration of his answer". Thomas Beddoes, rebelling against the argumentative and abstract methods of his own day, wrote : "If by detached figures I could show the truth of any proposition in an instant, I am forbidden, because this is an unmathematical mode of proceeding : that is, mathematical reasoning is supposed to be something independent of experience, and the science to be more refined than the experimental sciences. Hence, if a Greek writer happens to have written a demonstration a mile long, which demonstration can be nothing but a concatenation of the results of observation and experiment, I must take this tedious round, rather than be allowed to

arrive at the point desired by only traversing half a dozen yards, provided this shorter road leads through the unhallowed region of the senses."

When, in Stage C, the logical structure of geometry is to be considered, and detailed proofs are to be investigated, the Report states "The work . . . should not be learned by the boy with a view to its reproduction in examinations. It may be described as work in appreciation of the nature of geometry. It is work for class discussion." Thomas Beddoes does not hesitate to condemn 'learning by rote'. "But, according to the modern practice of education, instead of suffering children to follow the active tendency of their natures, or gently directing it, we forcibly debar them from the exercise of the senses, and condemn them to the horrible drudgery of learning by rote, the conceits of a tribe of sophists and semi-barbarians, to whom it is no reproach not to have entertained just ideas either concerning words or things. Next to actual blindfolding and muffling, to oblige children to learn the terms in which these conceits are couched is the happiest contrivance imaginable for keeping their minds unfurnished. . . ."

Again he recommends the use of models in solid geometry. Our Report states "The technical terms should be learnt at first by examination of solids and there should be a course of modelling with paper before formal instruction is begun". Beddoes writes: "To solid geometry we do not come anything near so well prepared by observation as to plane. The difficulty of imagining (which always depends on the want of opportunity or of power to perceive) the intersections of solids, is always very sensibly felt. And here it is almost as necessary as in mechanics, to exhibit the objects, whose qualities are to be taught; and to call in the joint assistance of the hands and eyes."

Both Thomas Beddoes and our Report consider the teaching of elementary geometry to children, both consider the possibility of extending the elementary ideas. He is careful to point out that "The more distinct and deep the impressions of the sense are at the beginning, the greater will the power of abstraction afterwards be, when the progress of his studies shall have carried him into the higher mathematics".

It may, however, encourage those who are depressed by the slow progress of ideas and who feel that we have not made as much progress as we should towards implementing Beddoes's ideas, to realise how far we have come from the accepted teaching of mathematics in 1769. I quote from *Mathematical Essays; or a New Introduction to the Mathematics being Essays on Vulgar and decimal Arithmetic with a General Preface on the Usefulness of Mathematical Learning*, by Benjamin Donn, Master of the Mathematical Academy at Bristol. Printed in 1769.

"Chapter XXX
"Of Alligation Medial.

"Alligation, (from the Verb Alligo, Latin) is the Rule, in which we show the method of solving Questions, relating to the Mixing of several simples, of different Prices or Qualities. It is divided into two Parts, Medial and Alternate.

"Alligation Medial (which is that we shall treat of in this chapter) is, when having given the several Quantities of the several Simples, and the Respective Rate (i.e. the Price or Quality) we are to find the Mean Rate of the Compound. And this, it is evident, may be found by the following Rule.

"The Rule. Multiply each Quantity by its respective Rate, and find the Sum of these Products (which will be the Value of the whole Mixture; therefore) now say, by the Golden Rule, as the whole Quantity is to the Sum of the Products, so is any given Quantity to its Rate—Two Examples will be sufficient to illustrate this Rule.

Question 1. Admit a Grocer would mix 10 lb. of Currants at 6d. per lb. with 12 lb. at 4d. per lb. and 14 lb. at 5d. per lb.; what is the value of 1 lb. of the Mixture." (Then follows the solution.)

" Chapter XXXI
" Of Alligation Alternate.

" Alligation Alternate is, when several things of different Prices are now to be mixed together, and it is required to find what Quantity may be taken of each Sort, so as the Mixture may be sold at a given Rate, without either Loss or Gain—In the common Method of working this Rule (which we shall now explain) there are three Varieties or Cases." (Then follow seven pages of instruction.) D. M. MILTON.

2454. *A problem in linear differential equations.*

A favourite theme on which examiners have worked many variations in the past twenty-five years is the derivation of the addition formula for some function from a differential equation. For instance, given that $f(0)=1$, $f'(x)=f(x)$, we deduce from

$$D_x f(x)f(a-x)=0$$

that

$$f(x)f(a-x)=f(a),$$

and therefore (taking $x+y$ for a),

$$f(x+y)=f(x)\,.\,f(y).$$

Similarly, (without of course introducing the circular functions) from the conditions $f(0)=0$, $f'(x)=g(x)$; $g(0)=1$, $g'(x)=-f(x)$, we readily prove the addition formulae for $f(x)$ and $g(x)$; for,

$$D_x\{f(x)g(a-x)+g(x)f(a-x)\}=0$$

and so giving the values 0, x in turn, and substituting $x+y$ for a we have

$$f(x+y)=f(x)g(y)+g(x)f(y).$$

In exactly the same way the addition formula for the function $g(x)$ is obtained by showing that $g(x)g(a-x)-f(x)f(a-x)$ is independent of x.

If instead of the first order equations $f'(x)=g(x)$, $g'(x)=-f(x)$, we take the second order equations

$$f''(x)+f(x)=0, \qquad g''(x)+g(x)=0,$$

together with the initial conditions $f(0)=g'(0)=0$, $g(0)=f'(0)=1$, the problem grows a little more difficult. To regain the relations

$$f'(x)=g(x), \qquad g'(x)=-f(x),$$

let us consider the functions $F(x)=g(x)-f'(x)$, $G(x)=g'(x)+f(x)$ which satisfy the conditions

$$F''(x)+F(x)=0, \qquad G''(x)+G(x)=0,$$
$$F(0)=F'(0)=G(0)=G'(0)=0.$$

If we appeal to the general theorem on the uniqueness of the solution of the differential equation $F''(x)+F(x)=0$ with the initial conditions

$$F(0)=F'(0)=0,$$

then it follows that $F(x)=0$ and $G(x)=0$, and the problem is solved. This,

however, assumes a deeper result than the one we are seeking to prove, and the question arises whether we cannot prove that the conditions

$$F''(x)+F(x)=0 \quad \text{and} \quad F(0)=F'(0)=0$$

imply $F(x)=0$, at a more elementary level. We observe in the first place that this result is not just a trivial consequence of Maclaurin's theorem ; for although $F^n(0)=0$ for any n, we do not yet know that the real function $F(x)$ is equal to its Maclaurin expansion. In fact, the condition that $F^n(0)=0$ for any n holds for the function defined by the equations

$$F(0)=0, \qquad F(x)=\exp(-1/x^2), \quad x\neq 0,$$

as may easily be verified. We can however readily show that $F^n(x)$ is bounded in any closed interval, uniformly in n, for if M is the greater of the upper bounds of the two continuous functions $|F(x)|$, $|F'(x)|$ in the interval $(-X, X)$, then, since $|F^n(x)|$ has one of the values $|F(x)|$, $|F'(x)|$ for any n, it follows that $|F^n(x)|<M$ for any x in $(-X, X)$ and any n. By Maclaurin's theorem there is a θ between 0 and 1 such that for $|x|\leqslant X$,

$$\left|F(x)\right| = \left|\frac{x^n}{n!}F^n(\theta x)\right| < \frac{MX^n}{n!} \to 0\ ,$$

so that $F(x)=0$ for $|x|\leqslant X$, and therefore for any x, since X is arbitrary.

But a much simpler proof is possible, for

$$D_x\{F^2+F'^2\}=2F'(F''+F)=0$$

and so $$F^2+F'^2=0,$$

whence $$F(x)=0.$$

The former of these arguments is immediately applicable to the general case in which

$$F^n(x)=\sum_{r=0}^{n-1} a_r F^r(x)$$

and $$F^r(0)=0, \quad 0\leqslant r\leqslant n-1.$$

For if $$k=1+\sum_{r=0}^{n-1}|a_r|, \quad \text{and} \quad \mu_r=\max_{0\leqslant p\leqslant r}|F^p(x)|, \quad |x|\leqslant X,$$

then, for $r\geqslant n$, $\mu_{r+1}\leqslant k\mu_r$ so that $\mu_{r+n}\leqslant k^r\mu_n$; it follows, therefore, as above, that

$$\left|F(x)\right| = \left|\frac{x^N}{N!}F^{(N)}(\theta x)\right| < \mu_n(kX)^N/N!\to 0 \text{ as } N\to\infty\,.$$

An alternative proof runs as follows. If we write the differential equation in the form

$$L_n(D)F(x)=0,$$

where $L(D)$ is a polynomial in D of degree n, and assume that the theorem holds for $n=1$ and $n=2$ then its truth for a general n follows by induction. For, if the theorem has been proved (on this assumption) for $n\leqslant N$, and if $L_{N+1}(D)$ has a linear factor $D-a$, and $L_{N+1}(D)=(D-a)L_N(D)$, then from

$$(D-a)\{L_N(D)F(x)\}=0 \quad \text{and} \quad \{L_N(D)F(x)\}_{x=0}=0\,.$$

follows, in turn, (by hypothesis)

$$L_N(D)F(x)=0$$

and thence $$F(x)=0\,;$$

and similarly, if $L_{N+1}(D)$ has a quadratic factor $Q(D)$, we deduce that $F(x)=0$. It remains to prove the theorem only for the cases $n=1$ and $n=2$. For $n=1$, we have to show that $F(x)=0$ identically if

$$F'(x)=aF(x) \qquad \text{and} \qquad F(0)=0,$$

and this is proved by observing that

$$D_x\{F(x)F(c-x)\}=0$$

so that

$$F(x)F(c-x)=0,$$

and so, taking $c=2x$, $\{F(x)\}^2=0$ and therefore $F(x)=0$. It remains only to deduce $F(x)=0$ from the conditions

$$F''(x)+bF'(x)+aF(x)=0, \qquad F(0)=F'(0)=0,$$

where $4a>b^2$ (so that D^2+bD+a has no simple factors), and this follows immediately from the fact that, since

$$D_x(aF^2+bFF'+F'^2)=-b(aF^2+bFF'+F'^2)$$

and

$$aF^2+bFF'+F'^2=0 \qquad \text{at } x=0,$$

therefore by the case $n=1$, we have

$$aF^2+bFF'+F'^2=0,$$

that is,

$$(F'+\tfrac{1}{2}bF)^2+\tfrac{1}{4}(4a-b^2)F^2=0,$$

whence

$$F(x)=0.$$

R. L. Goodstein.

2455. *Hexagone associé à un triangle.*

Soient un triangle $T\equiv ABC$, inscrit à un cercle (O, R) ; $a, b, c, 2p$ et r_a, r_b, r_c les mesures des côtés BC, CA, AB, du périmètre et des rayons des cercles exinscrits (I_a), (I_b), (I_c) ; H_a, H_a', H_b, H_b', H_c, H_c' les orthocentres des triangles I_aAB, I_aCA, I_bBC, I_bAB, I_cCA, I_cBC.

1. *Les côtés de l'hexagone $H_aH_a'H_bH_b'H_cH_c'$ sont équipollents à ceux des triangles T et $I_aI_bI_c$.*

Il est clair, en effet, que les triangles BI_aC, CI_bA, AI_cB sont équipollents aux triangles $H_b'AH_c$, $H_c'BH_a$, CH_bH_a' et que les quadrangles BCH_bH_c' et CBI_bI_c, CAH_cH_a' et ACI_cI_a, ABH_aH_b' et BAI_aI_b sont symétriques par rapport aux milieux des côtés BC, CA, AB de T.

2. *Les côtés $H_b'H_c$, H_aH_c', H_bH_a' de l'hexagone sont équidistants de l'orthocentre H du triangle T.*

Car, des équipollences invoquées (§ 1), il résulte que les distances des côtés en cause au point H sont égales à

$$HA+r_a=2R\cos A+r_a=2R+r=HB+r_b=HC+r_c.$$

3. *Les médiatrices des côtés BC, CA, AB et les hauteurs AA', BB', CC' du triangle T bissectent respectivement les côtés $H_c'H_b$, $H_a'H_c$, $H_b'H_a$ et les diagonales principales H_aH_a', H_bH_b', H_cH_c' de l'hexagone.*

Il suffira de démontrer que la médiatrice de BC passe au milieu de $H_c'H_b$ et que les points H_c', H_b sont équidistants de la hauteur BB'. Or, les hauteurs I_cH_c', I_bH_b des triangles I_cBC, I_bBC sont équidistante de la médiatrice de BC. De plus, les points H_c', H_b se projettent orthogonalement sur la droite BC en ses points de contact avec les cercles (I_c), (I_b) ; si X, Y désignent les projections orthogonales de H_b, H_b' sur la hauteur BB', on obtient, d'abord,

$$\cos\tfrac{1}{2}C=H_bX:H_bB=H_bX:CI_c,$$

puis

$$H_bX = CI_c \cos \tfrac{1}{2}C = p = H_b'Y.$$

4. *L'aire de l'hexagone équivaut à celle du triangle T augmentée de trois fois l'aire du triangle $I_aI_bI_c$.*

L'aire S de l'hexagone équivaut, en grandeur et en signe, à la somme

$$ABC + AH_cH_b' + BH_aH_c' + CH_bH_a' + BH_c'H_bC + CH_a'H_bA + AH_b'H_aB \quad (1)$$

des aires des triangles et des quadrangles qui la composent. Or,

$$\Sigma\, AH_cH_b' = \Sigma\, BI_aC = I_aI_bI_c - ABC\,; \quad \text{.....................(2)}$$

d'autre part, en raison des symétries invoquées, (§ 1),

$$BH_c'H_bC = CI_bI_cB = I_aI_bI_c - BI_aC,$$

$$\Sigma\, BH_c'H_bC = 3I_aI_bI_c - \Sigma\, BI_aC = 2I_aI_bI_c + ABC. \quad \text{...............(3)}$$

En définitive, $S = ABC + 3I_aI_bI_c$,

d'après les relations (1), (2), (3).

N.B. Nous laissons au lecteur le soin d'ajouter d'autres propriétés à cette curieuse figure.

V. Thébault.

2456. *Angle et points de Brocard d'un triangle.*

Cette note signale quelques constructions simples des éléments indiqués V et Ω, Ω' d'un triangle $T \equiv ABC$, connaissant le cercle circonscrit (O, R) et le point de Lemoine K de celui-ci.

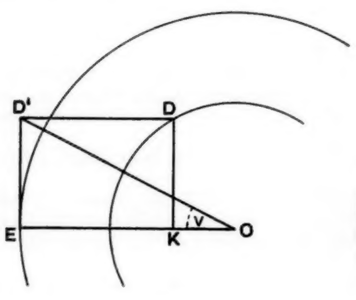

Fig. 1.

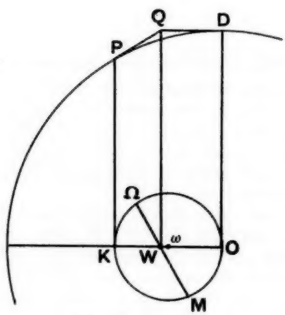

Fig. 2.

1. *Angle V* (Fig. 1). Soient KD la demi-corde du cercle (O, R) perpendiculaire à la droite OK qui rencontre le cercle $(O, R\sqrt{3})$ en E ; la parallèle à OK menée par D coupe la tangente en E au cercle $(O, R\sqrt{3})$ en D'. *L'angle (OD', OE) est égal à V*, car

$$R^2 - OK^2 = 3R^2/\cot^2 V = KD^2 = ED'^2.$$

2. *Points Ω, Ω'* (Fig. 2). Les tangentes au cercle (O, R) aux extrémités P, D des demi-cordes KP, OD perpendiculaires à OK se rencontrent en un point Q dont la projection orthogonale sur OK est W (centre isodynamique de T). La corde du cercle (ω) décrit sur OK comme diamètre (cercle de Brocard) menée par W et par le point M tel que $OM = \frac{1}{2}OK$ recoupe le cercle (ω) au point de Brocard Ω dont le symétrique par rapport à OK coïncide avec l'autre point de Brocard Ω'.

V. Thébault.

2457. *A congruence property of the terms of a series.*

Let us write

$$N_r = 1^r - \frac{r_2}{2\,.\,1!} + \frac{r_4}{2^2\,.\,2!} - \frac{r_6}{2^3\,.\,3!} + \ldots \pm \frac{r_k}{2^{k/2}(k/2)!},$$

where r is a positive integer or zero, r_k denotes $r(r-1)(r-2)\ldots$ to k factors and k is $r-1$ or r according as to whether r is odd or even.

The first number is $N_0 = 1$, and the first fourteen such numbers are 1, 1, 0, -2, -2, 6, 16, -20, -132, 28, 1216, 936, -12440 and -116072.

If r is odd, then, without loss of generality,

$$N_r = 1 - \frac{r(r-1)}{2\,.\,1!} + \frac{r(r-1)(r-2)(r-3)}{2^2\,.\,2!} - \frac{r(r-1)(r-2)(r-3)(r-4)(r-5)}{2^3\,.\,3!}$$
$$+ \ldots \pm \frac{r(r-1)\ldots 2}{2^{(r-1)/2}\,.\left(\dfrac{r-1}{2}\right)!},$$

$$N_{r-1} = 1 - \frac{(r-1)(r-2)}{2\,.\,1!} + \frac{(r-1)(r-2)(r-3)(r-4)}{2^2\,.\,2!}$$
$$- \frac{(r-1)(r-2)(r-3)r-4)(r-5)(r-6)}{2^3\,.\,3!} + \ldots \pm \frac{(r-1)(r-2)\ldots 2}{2^{(r-1)/2}\left(\dfrac{r-1}{2}\right)!};$$

and hence

$$N_r - N_{r-1} = -(r-1) + \frac{(r-1)(r-2)(r-3)}{2\,.\,1!} - \frac{(r-1)(r-2)(r-3)(r-4)(r-5)}{2^2\,.\,2!}$$
$$+ \ldots \mp \frac{(r-1)(r-1)(r-2)\ldots 2}{2^{(r-1)/2}\left(\dfrac{r-1}{2}\right)!}.$$

We can write the last term as

$$\frac{2(\overline{\tfrac{1}{2}r-1})(r-1)(r-2)\ldots 2}{2^{(r-1)/2}\left(\dfrac{r-1}{2}\right)!} = \frac{(r-1)(r-2)\ldots 2}{2^{(r-3)/2}\left(\dfrac{r-3}{2}\right)!}.$$

Thus

$$N_r - N_{r-1} = -(r-1) + \frac{(r-1)(r-2)(r-3)}{2\,.\,1!} - \ldots \mp \frac{(r-1)(r-2)\ldots 2}{2^{(r-3)/2}\left(\dfrac{r-3}{2}\right)!}$$
$$= -(r-1)N_{r-2}.$$

Thus we obtain the recurrence relation

$$N_r - N_{r-1} + (r-1)N_{r-2} = 0.$$

Theorem. For $r > 3$, $N_r \equiv 0 \pmod{r-2}$.

We have

$$N_r = N_{r-1} - rN_{r-2} + N_{r-2},$$

and similarly,

$$N_{r-1} = N_{r-2} - rN_{r-3} + 2N_{r-3}.$$

Thus $N_r = N_{r-2} - rN_{r-3} + 2N_{r-3} - rN_{r-2} + N_{r-2}$

$$= 2N_{r-2} + 2N_{r-3} - r(N_{r-2} + N_{r-3})$$

$$= -(N_{r-2} + N_{r-3})(r-2),$$

that is, $N_r \equiv 0 \pmod{r-2}$. MAX RUMNEY.

2458. *Geometric proof of an inequality.*

The inequality, $(m > 1)$,

$$\frac{(a+b+c+\ldots+k)^m}{n} > \frac{a^m+b^m+c^m+\ldots+k^m}{n}$$

is an example of convexity, and is discussed in this light in Hardy, Littlewood and Polya, *Inequalities*. I have found that a diagram makes the matter interesting to pupils.

Suppose the numbers $a, b, c, \ldots$ are arranged in ascending order, and draw the graph (Fig. 1) of $y = x^m$. If $m > 1$, the curve is convex to OX, so that if any two points on the curve are joined, the chord lies above the curve.

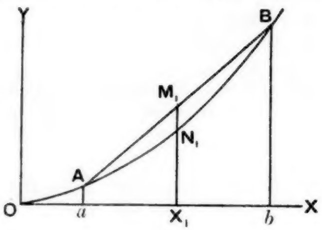

FIG. 1.

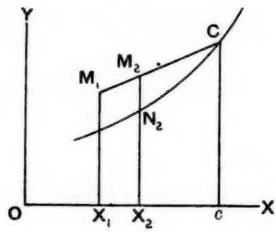

FIG. 2.

(i) Take the points $A\ (a, a^m)$ and $B\ (b, b^m)$ on the curve. Join AB and bisect at M_1. The mid-ordinate M_1X_1 of AB will meet the curve at a point N_1 *below* M_1, so that $M_1X_1 > N_1X_1$. Now $OX_1 = \frac{1}{2}(a+b)$, and so $N_1X_1 = \{\frac{1}{2}(a+b)\}^m$. Also M_1X_1 is the average of the ordinates at A and B and is equal to

$$\tfrac{1}{2}(a^m + b^m).$$

Thus it follows that

$$\frac{a^m+b^m}{2} > \left(\frac{a+b}{2}\right)^m.$$

(ii) In Fig. 1, insert C to the right of B, where C is the point (c, c^m), and divide M_1C in the ratio 1 : 2 (Fig. 2). Draw the ordinate M_2X_2 meeting the curve at N_2. The coordinates of M_1 are $\frac{1}{2}(a+b)$, $\frac{1}{2}(a^m+b^m)$, and those of C are (c, c^m). Hence by the usual ratio formula, $(\lambda q + \eta p)/(\lambda + \eta)$, where $\lambda = 1$ and $\eta = 2$, the coordinates of M_2 are $OX_2 = \frac{1}{3}(a+b+c)$, $M_2X_2 = \frac{1}{3}(a^m+b^m+c^m)$, and so $N_2X_2 = \{\frac{1}{3}(a+b+c)\}^m$, since N_2 is on the curve. Thus, as M_2 lies above N_2,

$$\frac{a^m+b^m+c^m}{3} > \left(\frac{a+b+c}{3}\right)^m.$$

(iii) If we now add the point D and divide MD at M_3 in the ratio 1 : 3, the same argument gives the inequality for $n = 4$, and by repetition of the process we arrive at the general result.

(iv) The postulates involved in the above proof are simple, namely (a) the

curve $y=x^m$ is convex to OX, (*b*) " above " is equivalent to " greater than ". A similar argument, reversing the inequality, may be used for $m<1$.

A little modification of the proof is required when some of the numbers $a, b, c, \ldots$ are equal.

J. GAGAN.

2459. *A diagram to illustrate the geometric series.*

Suppose $r<1$. On OX take $OP=r$, and $OQ=1$. Draw ordinates PL and QM. Take any point A on QM, and join OA cutting PL at B. Draw BA_1 parallel to OX, and let $AA_1=a$. The successive terms of the series, $a, ar, ar^2, \ldots$ may then be represented as follows.

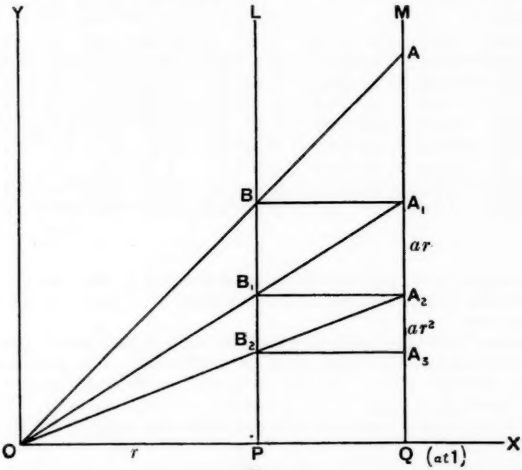

FIG.

Join OA_1 cutting PL at B_1, and draw B_1A_2 parallel to OX. Then $A_1A_2=ar$.

Join OA_2, cutting PL at B_2, and draw B_2A_3 parallel to OX. Then $A_2A_3=ar^2$.

The construction can now be repeated to represent $ar^3, ar^4 \ldots$. The similarity of the triangles formed may be used to show that the intercepts AA_1, A_1A_2, A_2A_3 represent the terms of the geometric series.

It can be seen that AQ is the sum to infinity, and since

$$AQ/OQ=AA_1/A_1B,$$

we have that the sum to infinity is $a/(1-r)$. The sum to n terms may also be derived.

The diagram may be compared with one formed by taking $r=OP>1$.

J. GAGAN.

2460. *A model of a developable surface, showing its edge of regression.*

The surface in question is that generated by the tangents to the helix whose parametric equations are

$$x=\tfrac{1}{2}\cos\theta, \qquad y=\tfrac{1}{2}\sin\theta, \qquad z=\tfrac{1}{2}\theta.$$

These tangents have the equations

$$\frac{x-\frac{1}{2}\cos\theta}{-\sin\theta}=\frac{y-\frac{1}{2}\sin\theta}{\cos\theta}=\frac{z-\frac{1}{2}\theta}{1}$$

The sections of the surface by the planes $z=3$, $z=-3$ are spiral curves whose equations are respectively

$$x=\tfrac{1}{2}\cos\theta-(3-\tfrac{1}{2}\theta)\sin\theta,\qquad y=\tfrac{1}{2}\sin\theta+(3-\tfrac{1}{2}\theta)\cos\theta,\qquad z=3$$

and $$x=\tfrac{1}{2}\cos\theta+(3+\tfrac{1}{2}\theta)\sin\theta,\qquad y=\tfrac{1}{2}\sin\theta-(3+\tfrac{1}{2}\theta)\cos\theta,\qquad z=-3.$$

The following table gives the values of x and y for the points of the first spiral for values of θ from $-180°$ to $180°$, at intervals of $10°$.

θ	x	y	θ	x	y	θ	x	y	θ	x	y
−180	−0·50	−4·57	−90	3·79	−0·50	0	0·50	3·00	90	−2·21	0·50
−170	0·28	−4·50	−80	3·73	0·15	10	−0·01	2·96	100	−2·18	0·12
−160	1·03	−4·30	−70	3·56	0·76	20	−0·50	2·83	110	−2·09	−0·23
−150	1·72	−3·98	−60	3·30	1·33	30	−0·94	2·62	120	−1·94	−0·54
−140	2·33	−3·56	−50	2·95	1·82	40	−1·32	2·35	130	−1·75	−0·81
−130	2·84	−3·04	−40	2·54	2·24	50	−1·64	2·03	140	−1·53	−1·04
−120	3·25	−2·46	−30	2·06	2·57	60	−1·89	1·67	150	−1·28	−1·21
−110	3·55	−1·82	−20	1·56	2·81	70	−2·07	1·29	160	−1·02	−1·34
−100	3·73	−1·16	−10	1·03	2·95	80	−2·18	0·89	170	−0·76	−1·41
									180	−0·50	−1·43

The second spiral is simply the reflection of the first in the x-axis, points corresponding in the reflection arising from values of θ equal in magnitude but opposite in sign.

The points of the first spiral are plotted on one-inch graph paper which is then laid on a rectangular sheet of stout cardboard. The points of the spiral are pricked through with a needle on to the cardboard, the position of the origin and axes also being carefully marked in the same way. The resulting holes in the cardboard are then labelled with the values of θ to which they correspond.

The second spiral is pricked out on a second sheet of cardboard of the same size as the first, care being taken to ensure that when one sheet is placed over the other the origins and axes in the two sheets coincide.

The first sheet is rigidly attached to the second, so that the two sheets lie in parallel planes 6 inches apart, the origin in the first sheet vertically above the origin in the second and the axes parallel. A Meccano set is convenient, if available, otherwise a light wooden frame may serve.

Lengths of strong cotton are now threaded through correspondingly numbered holes in the two sheets, each thread being drawn tight to form the generators of the surface. The thread must not be too thick, or distortion will occur. The 180° generator is inserted first. Viewing the model from above, generator 170 passes beneath 180, and 160 and 150 beneath both of these. Generator 140 passes *over* 180 and under the next three. Thereafter every generator passes over all generators previously inserted except the three immediately preceding it.

With a little care it will be found that a model results which clearly exhibits the relationship between a developable surface and its edge of regression.

J. G. Brennan.

2461. *Vectors in plane kinematics.*

The following examples show how some of the properties of rolling curves can be conveniently deduced from two or three vector formulae.

If P describes a curve of arc s and if $\mathbf{r}$ (x, y, z) is the position vector of P referred to a fixed origin O at time t, then

$$\dot{\mathbf{r}} = v\mathbf{t}, \qquad \dot{\mathbf{t}} = \kappa v \mathbf{n}, \qquad \text{.............................(1), (2)}$$

where $\mathbf{t}$ is the unit tangent in the direction of s increasing, $\mathbf{n}$ is the unit (principal) normal, κ is the curvature of the path at P and v is the speed of P. In plane applications $\mathbf{r} \equiv (x, y)$ and the set $\mathbf{t}$, $\mathbf{n}$, $\mathbf{k}$ (where $\mathbf{k}$ is the unit vector perpendicular to the plane) is chosen to be a right-handed system.

We have also the formula giving the absolute rate of change of $\mathbf{r}$ in terms of its rate of change relative to a frame rotating with angular velocity $\boldsymbol{\omega}$, namely

$$\dot{\mathbf{r}} = \partial \mathbf{r}/\partial t + \boldsymbol{\omega} \times \mathbf{r}. \qquad \text{.....................................(3)}$$

The partial derivative sign indicates rate of change relative to the moving frame and in plane applications we have $\boldsymbol{\omega} = \omega \mathbf{k}$; if $\mathbf{k}$ is directed above the plane, the angular velocity is ω counter-clockwise.

Let a curve (or lamina) C_b roll on a curve C_s fixed in space. Let P and P' (Fig. 1) coincide at $t = 0$; then we *define rolling by the equality of the arcs IP and IP′*.

1. *When a lamina rolls on a curve in its plane the point of the lamina in contact with the curve is instantaneously at rest.*

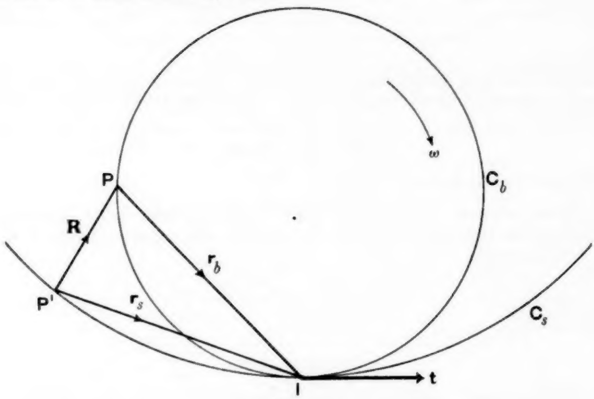

Fig. 1.

In the diagram, P' is a fixed origin and P is the origin of a frame rotating with angular velocity $-\omega\mathbf{k}$. By (1)

$$\dot{\mathbf{r}}_s = v\mathbf{t} = \partial \mathbf{r}_b/\partial t,$$

where v is the speed of I along C_s or C_b and $\mathbf{t}$ is the common unit tangent at I. Also, by (3),

$$\dot{\mathbf{r}}_b = \partial \mathbf{r}_b/\partial t + (-\omega\mathbf{k}) \times \mathbf{r}_b.$$

Again,

$$\mathbf{R} = \mathbf{r}_s - \mathbf{r}_b, \qquad \dot{\mathbf{R}} = \dot{\mathbf{r}}_s - \dot{\mathbf{r}}_b.$$

Thus

$$\dot{\mathbf{R}} = \omega\mathbf{k} \times \mathbf{r}_b\,;$$

that is, P is moving with speed $\omega \,.\, IP$ in the direction perpendicular to IP. It follows from the rigidity of the lamina that *any* point Q is moving with speed $\omega \,.\, IQ$ in a direction perpendicular to IQ and that the speed of the point I of the lamina is zero.

2. *When a lamina moves in any manner in its own plane, the body centrode rolls on the space centrode.*

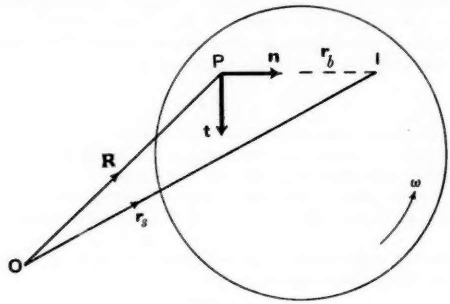

FIG. 2.

Let a point P of a lamina have velocity $v\mathbf{t}$, when the angular velocity of the lamina is ω counter-clockwise ($\boldsymbol{\omega} = \omega\mathbf{k}$). Then the point of zero velocity is I such that

$$\mathbf{PI} = \mathbf{r}_b = (v/\omega)\mathbf{n},$$

where $\mathbf{t}$, $\mathbf{n}$, $\mathbf{k}$ are as defined above.

As motion proceeds, $\mathbf{r}_s$ gives the space-centrode, referred to the fixed origin O, while $\mathbf{r}_b$ gives the body centrode referred to the rotating frame $\mathbf{t}$, $\mathbf{n}$, $\mathbf{k}$ with origin P. We have

$$\dot{\mathbf{R}} = v\mathbf{t}, \qquad \dot{\mathbf{r}}_s = v_s\mathbf{t}_s,$$

and, using (3),

$$\dot{\mathbf{r}}_b = v_b\mathbf{t}_b + \omega\mathbf{k} \times \mathbf{r}_b,$$

where suffixes b and s refer to body and space loci of I respectively.

Also $$\dot{\mathbf{R}} = \dot{\mathbf{r}}_s - \dot{\mathbf{r}}_b,$$

so that

$$v\mathbf{t} = v_s\mathbf{t}_s - v_b\mathbf{t}_b - \omega\mathbf{k} \times (v/\omega)\mathbf{n},$$

or, since $$\mathbf{k} \times \mathbf{n} = -\mathbf{t},$$

$$v_s\mathbf{t}_s = v_b\mathbf{t}_b,$$

showing that the centrodes *touch* at I and that they are described with the same speed.

3. *When a lamina of curvature κ_b rolls with angular velocity ω on a fixed curve, in its plane, of curvature κ_s, the point of contact moves (or the centrodes are described) at speed v, where*

$$\omega = v(\kappa_b - \kappa_s).$$

Consider $\mathbf{t}$ as a unit tangent to (i) the fixed space centrode C_s, and (ii) the body centrode C_b referred to the rotating frame $\mathbf{t}$, $\mathbf{n}$, $\mathbf{k}$, rotating with angular velocity $-\omega\mathbf{k}$ (Fig. 1). Differentiating with respect to time,

$$\frac{d\mathbf{t}}{dt}=\frac{d\mathbf{t}}{ds'}\cdot\frac{ds'}{dt}=\frac{\partial\mathbf{t}}{\partial s}\cdot\frac{ds}{dt}+(-\omega\mathbf{k})\times\mathbf{t},$$

where s' is the arc of C_s and s is the corresponding (equal) arc of C_b. Hence, using (2),

$$\kappa_s v\mathbf{n}=\kappa_b v\mathbf{n}-\omega\mathbf{n},$$

whence

$$\omega=v(\kappa_b-\kappa_s).$$

J. E. C. Gliddon.

2462. *Single letters for angles.*

The usual notation of single letters for angles suffers from several disadvantages :

(1) Many pupils find it difficult to write a letter, with suffix, in the small space available.

(2) Either a Greek letter is used, which adds to their difficulties, or a Roman " small " letter, which cuts across the convention that small letters are used for lengths and capitals for angles.

(3) An angle already divided into two is referred to by a composite name, such as $\beta+\gamma$ or $\alpha_1+\alpha_2$. This is most unsatisfactory, as it is one angle and should have one name. Some riders are thus reduced almost to exercises in algebra. The alternative is to mix the two notations, and write, for example, (Fig. 1),

$$\angle BAC+\beta=\gamma_2,$$

but this is a makeshift compromise and looks it.

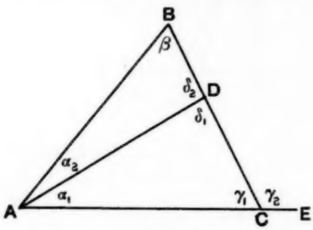

Fig. 1.

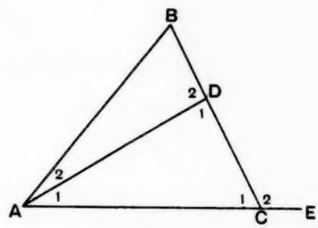

Fig. 2.

All these difficulties disappear if the notation illustrated in Fig. 2 is used. Thus

$$\angle A_1+\angle D_1=\angle C_2,$$

and

$$\angle BAC+\angle B=\angle C_2.$$

The figure is less crowded and the three-letter system mixes quite easily and naturally with the single letters.

This notation was suggested to me by a pupil, R. R. Northen.

E. H. Lockwood.

2463. *Maximum range of a projectile on any plane.*

The object of this note is to establish the theorem that if P is the point of contact of a trajectory and its enveloping parabola, then (a) P determines the point of striking when the range on the plane is a maximum, and (b) at P the projectile is moving at right angles to its direction of projection, and also to adapt this theorem to the more general problem.

1. The member of the family of trajectories

$$y = x \tan\theta - \tfrac{1}{2}(gx^2/V^2)\sec^2\theta$$

which meets the fixed plane whose line of greatest slope in the plane containing the point of projection O is

$$y = x\tan\alpha - p\sec\alpha$$

at the maximum distance from A (p cosec α, 0), meets that plane at the point where this trajectory touches its envelope

$$y + \tfrac{1}{2}\frac{gx^2}{V^2} = \frac{V^2}{2g}.$$

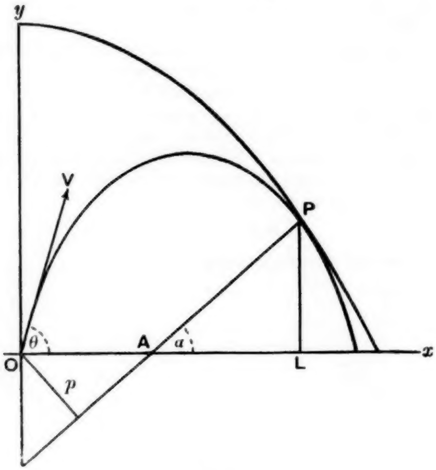

FIG.

Proof. If AP is a maximum, $PL = y$ is also a maximum. But

$$y - (y + p\sec\alpha)\tan\theta\cot\alpha + \tfrac{1}{2}(g/V^2)\sec^2\theta\,(y + p\sec\alpha)^2\cot^2\alpha = 0.$$

Since $dy/d\theta = 0$ for a maximum,

$$(y + p\sec\alpha)\cot\alpha\sec^2\theta\{-V^2 + g\tan\theta\,(y + p\sec\alpha)\cot\alpha\} = 0.$$

Thus $x\tan\theta = V^2/g$, whence $y = V^2/2g - gx^2/2V^2$.

Therefore the trajectory meets AP when AP is a maximum, at a point P on the enveloping parabola, and P is the point

$$\left\{\frac{V^2}{g}\cot\theta, \quad \frac{V^2}{2g}(1 - \cot^2\theta)\right\},$$

where $\cot\theta$ is the positive root of the equation

$$V^2(1-\cot^2\theta)=2V^2\tan\alpha\cot\theta-2gp\sec\alpha.$$

2. At this point, the gradient of the trajectory is

$$\tan\theta-(gx/V^2)\sec^2\theta=-\cot\theta.$$

Therefore the point P is a point on the trajectory where the direction of motion is perpendicular to the direction of projection. Even if AP is vertical, on striking at maximum height, the body is moving down!

3. Applying the above to the problem of finding the direction of projection and maximum range, on an inclined plane of inclination α to the horizontal, from a point O which is at a perpendicular distance p from the plane, we have that P is the point of intersection in the positive quadrant of the enveloping parabola

$$y+\frac{gx^2}{2V^2}=\frac{V^2}{2g}$$

and the line $y=x\tan\alpha-p\sec\alpha$.

If R is the required maximum range, $R\cos\alpha+p\operatorname{cosec}\alpha=x$. Therefore

$$x^2+\frac{2xV^2}{g}\tan\alpha-\frac{2pV^2}{g}\sec\alpha-\frac{V^4}{g^2}=0.$$

Taking the positive root,

$$x=\frac{V^2}{g}\left\{\left(\sec^2\alpha+\frac{2pg}{V^2}\sec\alpha\right)^{\frac{1}{2}}-\tan\alpha\right\}.$$

Therefore the maximum range is

$$R=\frac{V^2}{g}\sec^2\alpha\left\{\left(1+\frac{2pg\cos\alpha}{V^2}\right)^{\frac{1}{2}}-\sin\alpha-\frac{pg}{V^2}\cot\alpha\right\}.$$

At P the gradient of the envelope is

$$-gx/V^2=-\cot\theta,$$

where θ is the angle of elevation of projection. Thus the elevation of projection is

$$\cot^{-1}\left\{\sec\alpha\left(1+\frac{2pg\cos\alpha}{V^2}\right)^{\frac{1}{2}}-\tan\alpha\right\}.$$

4. *Special cases.*

If $p=0$, $R_{max}=V^2/g(1+\sin\alpha)$,

and $\cot\theta=\sec\alpha-\tan\alpha=\cot(\frac{1}{2}\alpha+\frac{1}{4}\pi)$,

that is, $\theta=\frac{1}{4}\pi+\frac{1}{2}\alpha$.

(ii) If $x=p$ and $\alpha=\frac{1}{2}\pi$, then $\cot\theta=gp/V^2$ and

$$R_{max}=\frac{V^2}{2g}\left(1-\frac{p^2g^2}{V^4}\right)=-p\cot 2\theta.$$

(iii) If $\alpha=0$, $\cot\theta=(1+2pg/V^2)^{\frac{1}{2}}$ and

$$R_{max}=x_{max}=\frac{V}{g}(V^2+2pg)^{\frac{1}{2}}.$$

A. J. L. Avery.

2464. *Some properties of the triangle.*

In any triangle ABC, we mark

L, M, N the midpoints of BC, CA, AB ;

E_0, E_1, E_2, E_3 the centres of the inscribed and escribed circles ;

P_0, P_1, P_2, P_3 the points of contact of these circles with BC ;

Q_0, Q_1, Q_2, Q_3 the points of contact of these circles with CA ;

R_0, R_1, R_2, R_3 the points of contact of these circles with AB.

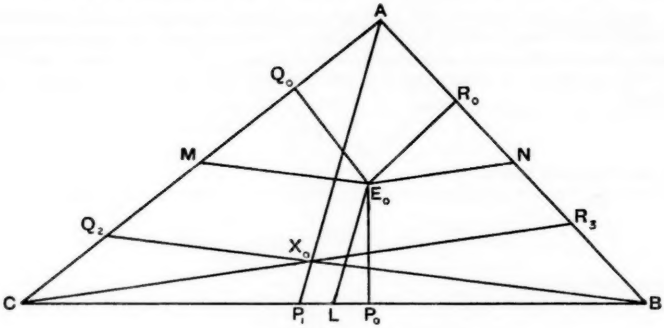

FIG. 1.

Then AP_1, BQ_2, CR_3 *are concurrent (in* X_0*) and are also parallel to* E_0L, E_0M, E_0N *respectively ; further, the areas of the triangles* X_0BC, X_0CA, X_0AB *are equal to the areas of the quadrilaterals* $E_0Q_0AR_0$, $E_0R_0BP_0$, $E_0P_0CQ_0$ *respectively.*

There are of course three corresponding theorems connected with E_1, E_2, E_3. For instance, AP_2, BQ_1, CR_0 are concurrent in X_3 and are parallel to E_3L, E_3M, E_3N respectively. Also the triangles X_3BC, X_3CA, X_3AB are equal in area to the quadrilaterals $E_3Q_3AR_3$, $E_3R_3BP_3$, $E_3P_3CQ_3$ respectively.

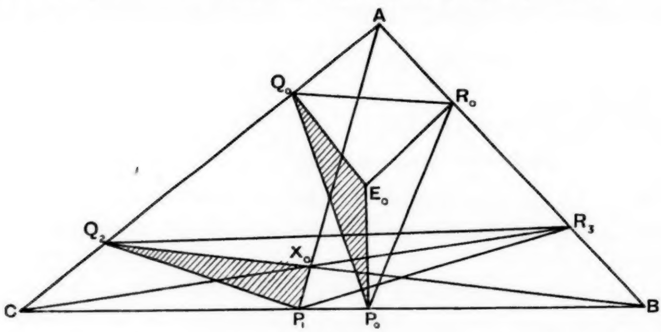

FIG. 2.

Also *the triangles* $X_0Q_2R_3$, $X_0R_3P_1$, $X_0P_1Q_2$ *are equal in area to the triangles* $E_0Q_0R_0$, $E_0R_0P_0$, $E_0P_0Q_0$ *respectively.*

Again, there are corresponding results for E_1, E_2, E_3. Thus, for example, the triangles $X_3Q_1R_0$, $X_3R_0P_2$, $X_3P_2Q_1$ are equal in area to the triangles $E_3Q_3R_3$, $E_3R_3P_3$, $E_3P_3Q_3$ respectively.

The reader is asked to supply his own proofs and further diagrams.

D. F. FERGUSON.

2465. *The three regular polyhedra obtained by stellating a regular dodecahedron.*

If a regular pentagon A is stellated to form a pentagram, if the corners of the pentagram are joined and the pentagon thus obtained is stellated, the figure shown is obtained.

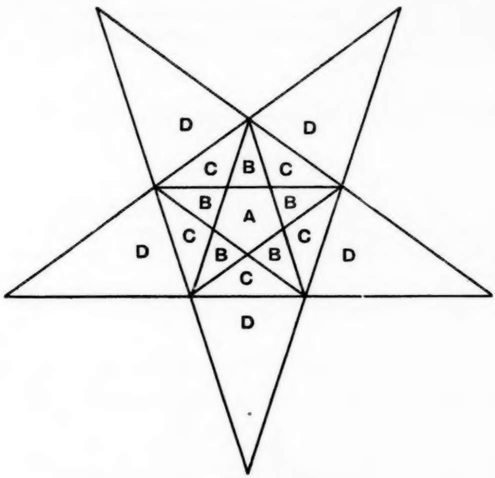

FIG.

12 of these figures contain the nets of:

the regular dodecahedron (12 of A);

the small stellated dodecahedron (60 of B arranged in 12 open pyramids of 5);

the great dodecahedron (60 of C arranged in 30 pairs);

the great stellated dodecahedron (60 of D arranged in 20 open pyramids of 3).

The construction of these four solids from 12 of the given figure is economical in material and in the time used for drawing, but has the disadvantage that every edge has to be formed by joining (instead of bending).

The figure can be used when a class of children is combining to build up each solid from the one before it. Solid instead of open pieces are easier to add and then more than 12 of the given figure are required, together with a previously constructed regular dodecahedron with faces equal to A.

Regular dodecahedron + 12 pyramids with base A and slant faces B = small stellated dodocahedron;

Small stellated dodecahedron + 30 solids with 2 faces C and 2 faces B = great dodecahedron;

Great dodecahedron + 20 double pyramids with 3 faces D and 3 faces C = great stellated dodecahedron.

The building up of the plane figure, starting from A, should be compared with the building up of the 3 dodecahedra by successive stellations of the given dodecahedron. To make the process of stellation clearer this dodecahedron should be painted in 6 different colours and the pieces added painted in the same 6 colours, in such a way that each face has the same colour as its extensions. Because D is large compared with A, the construction of the plane figure starting with A makes fairly severe demands on accuracy. Other methods may therefore be preferred for drawing the various pieces that each pupil will need for his contribution to the model.

Thin cardboard, adhesive tape and gloss paint have been used in constructing models of this type. With experienced pupils a quick-drying cement gives better results than adhesive tape. D. F. BEVIS WHITE.

2466. *Geometrical proof of the tangent-intercept property of the three-cusped hypocycloid.*

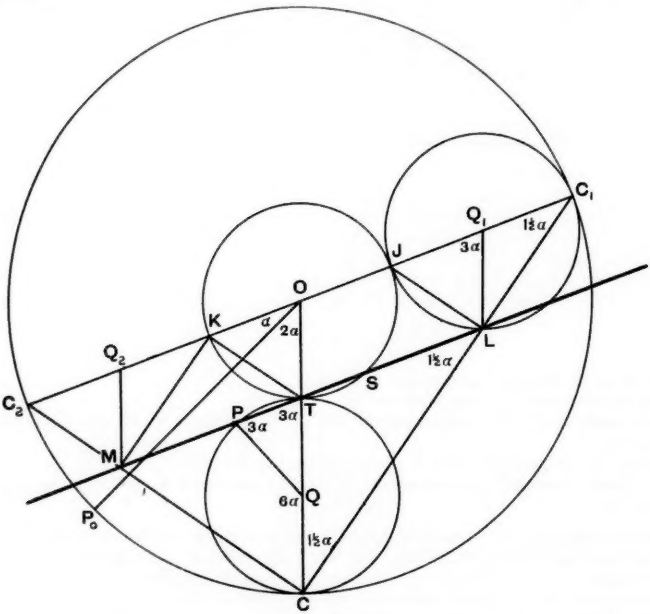

FIG. 66.
The heavy line $LTPM$ is the tangent to the hypocycloid.

The circle $C_1CP_0C_2$ has centre O and radius $3r$. On the inside of its circumference rolls a circle CPT of centre Q and radius r. P is that point of the rolling circle which started from P_0, C is the instantaneous centre of rotation

of the rolling circle and hence PC is the normal and PT the tangent to the locus of P (the three-cusped hypocycloid). The arc PC = arc P_0C; thus, if $\angle P_0OC = 2\alpha$, $\angle PQC = 6\alpha$, and hence $\angle QPT = \angle QTP = 3\alpha$.

Construction. From TP produced both ways, cut off TL and TM equal to $2r$. Produce CL and CM to meet the fixed circle at C_1 and C_2. Join C_1O and C_2O.

Proof. $\angle MCL$, in the semi-circle MCL, is a right angle, and so C_1, O, C_2 are collinear. $\angle TCL = \angle TLC = 1\frac{1}{2}\alpha$, $\angle OC_1C = \angle OCC_1 = 1\frac{1}{2}\alpha = \angle TLC$. Thus C_1C_2 is parallel to LM.

TL and KJ are equal and parallel, so that JL and KT are equal and parallel. But KT is parallel to C_2C; thus JL is parallel to C_2C, and hence $\angle JLC_1$ is a right angle. Thus L is on the circle on JC_1 as diameter.

In the isosceles triangle C_2OC,

$$\angle C_2 + \angle C_2CO = 2(\tfrac{1}{2}\pi - 1\tfrac{1}{2}\alpha) = \pi - 3\alpha.$$

Hence $\angle C_2OC = 3\alpha$, so that $\angle C_2OP_0 = \alpha$ and arc $P_0CC_1 = 3r(\pi - \alpha)$. Thus

$$\begin{aligned} 2\pi r + \text{arc } C_1L &= 2\pi r + r(\pi - 3\alpha) \\ &= 3r(\pi - \alpha) \\ &= \text{arc } P_0CC_1. \end{aligned}$$

Hence L is on the hypocycloid traced out by P. Similarly, M is on this hypocycloid. We have thus proved that the piece of a tangent to a three-cusped hypocycloid intercepted between its two intersections with the curve is of constant length $4r$.

This geometrical proof may be well-known: if so, I should welcome a reference.

W. HOPE-JONES.

2467. *A problem in statics.*

The following example appears in some textbooks* as an illustrative example in the chapter on friction.

"A heavy cubical block of edge $2a$ is placed on a rough table with one face parallel to the edge of the table and at a distance $a \cot \alpha$ from it; to the centre of this face a light smooth rod of length l is freely jointed; it passes over the smooth edge of the table and carries a weight W at its end. Show that as W is increased the equilibrium of the block is broken by its tilting about an edge if

$$\mu > \frac{l \cos \alpha \sin^2 \alpha}{a + l \sin \alpha \cos \alpha (\sin \alpha - \cos \alpha)}."$$

The interesting thing about it is that the solution assumes that the block will tip, if at all, about that lower edge nearest to the edge of the table, whereas, in fact, for certain values of α, the block will tip about the edge furthest from the edge of the table. Indeed, this is immediately obvious for very small values of α, as the action of the rod on the block is then approximately a force directed vertically upwards.

More precisely, the results are as follows:

(i) if $a/l > k_0$ (a certain constant) and α has any value greater than $\alpha_3 = \sin^{-1}(a/l)$, the only possibilities are sliding or tipping about the lower edge nearest to the edge of the table, with the discriminatory condition given in the original question;

* For instance, A. S. Ramsey, *Statics*, 2nd edition (Cambridge University Press, 1941), p. 141. Here the problem is marked [S], indicating that it came originally from a schol rship paper.

(ii) if, however, $a/l < k_0$, there are two angles α_1, α_2, where

$$0 < \alpha_3 < \alpha_1 < \alpha_2 < 45^\circ ,$$

and when α lies between α_3 and α_1 or between α_2 and 90°, the possibilities are as in case (i), but when α lies between α_1 and α_2 the tipping, if it occurs, will be about the lower edge furthest from the edge of the table. The condition for tipping is now

$$\mu > -\frac{l \cos \alpha \sin^2 \alpha}{a + l \sin \alpha \cos \alpha (\sin \alpha - \cos \alpha)},$$

the right-hand side of the original inequality for μ having now become negative.

The values α_1, α_2 lie one below and one above a critical value

$$\alpha_0 = \tfrac{1}{2}\sin^{-1}(\tfrac{2}{3}) = 20^\circ\ 54' \quad \text{approximately},$$

and the constant k_0 is $\frac{1}{3}(\cos \alpha_0 - \sin \alpha_0) = \cdot 1925$ approximately.

F. G. Maunsell.

2468. *Proof of a theorem on determinants.*

Let

$$\Delta = \begin{vmatrix} a & h & g & x \\ h & b & f & y \\ g & f & c & z \\ x & y & z & u \end{vmatrix}, \qquad D = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}.$$

The following, which I have not seen elsewhere, is a simple proof of the well-known theorem that Δ is a perfect square when $D = 0$.

Denoting co-factors of the elements in D by capital letters, we have

$$Aa + Hh + Gg = D = 0,$$
$$Ah + Hb + Gf = 0,$$
$$Ag + Hf + Gc = 0.$$

Then, on replacing row 1 in Δ by

$$A\ (\text{row } 1) + H\ (\text{row } 2) + G\ (\text{row } 3),$$

we obtain

$$A\Delta = \begin{vmatrix} 0 & 0 & 0 & Ax + Hy + Gz \\ h & b & f & y \\ g & f & c & z \\ x & y & z & u \end{vmatrix}$$

$$= -(Ax + Hy + Gz) \begin{vmatrix} h & b & f \\ g & f & c \\ x & y & z \end{vmatrix},$$

that is, $$A\Delta = -(Ax + Hy + Gz)^2;$$

similarly, $$B\Delta = -(Hx + By + Fz)^2,$$

and $$C\Delta = -(Gx + Fy + Cz)^2.$$

Thus $A = 0$ implies $H = G = 0$, etc., so that $A = B = C = 0$ implies $\Delta \equiv 0$, and in this case there is nothing to prove. But if A, B, C are not all zero, the required expression for Δ is given by (at least) one of the three equations above, and the theorem is proved.

If we write

$$\Delta = -(\alpha x + \beta y + \gamma z)^2,$$

it is easy to see that $\alpha^2 = A$, $\beta^2 = B$, $\gamma^2 = C$; but to determine α, β, γ with the least ambiguity possible (when $A \neq 0$, for example), we equate the three ratios A/α, H/β, G/γ to a definite one of the two values of $A^{\frac{1}{2}}$.

The above discussion is valid when the elements of Δ are complex numbers, and can be extended in an obvious way to apply to determinants of any order of the same type.

J. BURR.

2469. *On Note* 2263 (*Maxima and minima*).

Dr. Maxwell's paradox involves the invalid assumption that du/dr necessarily vanishes at an extremum. That the condition is neither necessary nor sufficient may be shown as follows.

Consider a function u of a variable r which is a function of s, the arc length along a given curve C. The condition for a stationary value of u on C is that $du/ds = 0$, that is,

$$\frac{du}{dr}\,\frac{dr}{ds} = 0.$$

In particular, a point at which $dr/ds = 0$ and du/dr exists is a stationary point. In Dr. Maxwell's problem we find that $dr/ds = 0$ where $\theta = \frac{1}{4}k\pi$, and du/dr exists at these points.

More generally, s_0 is a stationary point if

$$\lim_{s \to s_0} \frac{du}{dr}\,\frac{dr}{ds} = 0,$$

so that not only may du/dr be non-zero at s_0, it may even have an infinite discontinuity there. For example, if $u \equiv +(r-1)^{\frac{1}{2}}$ and C is the curve $r = 1 + \theta^4$ (where r, θ are polar coordinates), we see that $u = \theta^2$ on C, and u has a minimum value at $r = 1$, $\theta = 0$; although du/dr becomes infinite there.

On the other hand, if r is such a function of s that dr/ds has infinities, it is clear that the vanishing of du/dr does not alone ensure the existence of a stationary point. For example, if $u = r^3$ and $r = (s - s_0)^{\frac{1}{3}}$, there is no stationary point, although du/dr vanishes at s_0.

To see that Dr. Maxwell's first solution is correct, we should note that $d\theta/ds$ is bounded on C; to see that it is the complete solution, we should note that $d\theta/ds$ does not vanish on C.

J. BURR.

2470. *An introduction to quaternions.*

For the present-day applied mathematician, who early learns the use of vectors, it is possibly convenient to reverse historical evolution and to define quaternions in terms of real numbers and vectors. Thus, a quaternion $\bar{p}$ is a combination of a real number a and a real vector $\mathbf{x}$, which is written

$$\bar{p} = a + \mathbf{x}.$$

If λ, μ, ... are real numbers, we define addition and multiplication of quaternions by the equations

$$\pm\lambda(a+\mathbf{x}) \pm \mu(b+\mathbf{y}) \ldots = (\pm\lambda a \pm \mu b \ldots) + (\pm\lambda\mathbf{x} \pm \mu\mathbf{y} \ldots),$$

$$(a+\mathbf{x})(b+\mathbf{y}) = (ab - \mathbf{x}\cdot\mathbf{y}) + (a\mathbf{y} + b\mathbf{x} + \mathbf{x}_\wedge\mathbf{y}).$$

The second equation implies, in general, that

$$\bar{p}\bar{q} \neq \bar{q}\bar{p};$$

however, it is easy to verify the result

$$(\bar{p}\bar{q})\bar{r}=\bar{p}(\bar{q}\bar{r}),$$

so that we have the unambiguous product $\bar{p}\bar{q}\bar{r}$.

Quaternions provide an easy way of compounding finite rotations. Suppose the axis of rotation is taken as the z-axis, with an arbitrary point O on it as origin. Let P be a representative point with spherical co-latitude θ; the (z, x) plane is taken to pass through the initial position of P so that the initial vector OP is

$$\mathbf{x}=(r\sin\theta,\ 0,\ r\cos\theta).$$

For a rotation ϕ about the axis, the final vector OP then is

$$\mathbf{x}'=(r\sin\theta\cos\phi,\ r\sin\theta\sin\phi,\ r\cos\theta).$$

But, if $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are unit vectors parallel respectively to the axes of x, y, z, it is easy to verify that

$$(\cos\tfrac{1}{2}\phi+\mathbf{k}\sin\tfrac{1}{2}\phi)(\mathbf{i}r\sin\theta+\mathbf{k}r\cos\theta)(\cos\tfrac{1}{2}\phi-\mathbf{k}\sin\tfrac{1}{2}\phi)$$
$$=\mathbf{i}r\sin\theta\cos\phi+\mathbf{j}r\sin\theta\sin\phi+\mathbf{k}r\cos\theta,$$

that is,

$$(\cos\tfrac{1}{2}\phi+\mathbf{k}\sin\tfrac{1}{2}\phi)\mathbf{x}(\cos\tfrac{1}{2}\phi-\mathbf{k}\sin\tfrac{1}{2}\phi)=\mathbf{x}'.$$

Hence we obtain the general rule that a rotation ϕ about an axis parallel to the unit vector $\mathbf{k}$ transforms the vector $\mathbf{x}$ to the vector $\mathbf{x}'$, where $\mathbf{x}'$ is the quaternion product

$$\mathbf{x}'=(\cos\tfrac{1}{2}\phi+\mathbf{k}\sin\tfrac{1}{2}\phi)\mathbf{x}(\cos\tfrac{1}{2}\phi-\mathbf{k}\sin\tfrac{1}{2}\phi).$$

A second rotation ω about an axis parallel to the unit vector $\mathbf{l}$ changes $\mathbf{x}'$ to $\mathbf{x}''$, where

$$\mathbf{x}''=(\cos\tfrac{1}{2}\omega+\mathbf{l}\sin\tfrac{1}{2}\omega)\mathbf{x}'(\cos\tfrac{1}{2}\omega-\mathbf{l}\sin\tfrac{1}{2}\omega).$$

Now we have

$$(\cos\tfrac{1}{2}\omega+\mathbf{l}\sin\tfrac{1}{2}\omega)(\cos\tfrac{1}{2}\phi+\mathbf{k}\sin\tfrac{1}{2}\phi)=\cos\tfrac{1}{2}\theta+\mathbf{n}\sin\tfrac{1}{2}\theta,$$
$$(\cos\tfrac{1}{2}\phi-\mathbf{k}\sin\tfrac{1}{2}\phi)(\cos\tfrac{1}{2}\omega-\mathbf{l}\sin\tfrac{1}{2}\omega)=\cos\tfrac{1}{2}\theta-\mathbf{n}\sin\tfrac{1}{2}\theta,$$

where

$$\cos\tfrac{1}{2}\theta=\cos\tfrac{1}{2}\omega\cos\tfrac{1}{2}\phi-\mathbf{l}\cdot\mathbf{k}\sin\tfrac{1}{2}\omega\sin\tfrac{1}{2}\phi,$$
$$\mathbf{n}\sin\tfrac{1}{2}\theta=\mathbf{k}\cos\tfrac{1}{2}\omega\sin\tfrac{1}{2}\phi+\mathbf{l}\sin\tfrac{1}{2}\omega\cos\tfrac{1}{2}\phi+(\mathbf{l}\wedge\mathbf{k})\sin\tfrac{1}{2}\omega\sin\tfrac{1}{2}\phi.$$

Hence

$$\mathbf{x}''=(\cos\tfrac{1}{2}\theta+\mathbf{n}\sin\tfrac{1}{2}\theta)\mathbf{x}(\cos\tfrac{1}{2}\theta-\mathbf{n}\sin\tfrac{1}{2}\theta);$$

that is, the two rotations are equivalent to a single rotation θ about an axis parallel to $\mathbf{n}$.

Another physical process which can be represented by quaternions is that of reflection. Suppose $\mathbf{k}$ is an arbitrary unit vector, which is used to define the direction of the z-axis. Taking an arbitrary vector

$$\mathbf{x}=a\mathbf{i}+b\mathbf{j}+c\mathbf{k},$$

we easily find that

$$\mathbf{k}(a\mathbf{i}+b\mathbf{j}+c\mathbf{k})\mathbf{k}=(a\mathbf{i}+b\mathbf{j}-c\mathbf{k});$$

this vector is the reflection of the given vector $\mathbf{x}$ in a plane through the origin normal to $\mathbf{k}$. Suppose we consider the effect of successive reflections in planes through the origin perpendicular to unit vectors $\mathbf{k}, \mathbf{l}$. The vector $\mathbf{x}$ transforms to the vector

$$\mathbf{x}'=\mathbf{lkxkl}=(\mathbf{l}\cdot\mathbf{k}+\mathbf{k}\wedge\mathbf{l})\mathbf{x}(\mathbf{l}\cdot\mathbf{k}-\mathbf{k}\wedge\mathbf{l});$$

that is, the total effect is equivalent to a rotation, through twice the angle between the reflecting planes, about their line of intersection as axis. The application of this to images in two mirrors is obvious.

From these results, it is easy to obtain a number of well-known results, such as are given in the textbooks; for example, see Brand, *Vector and tensor analysis*, Chapter X.

I. A. EVANS.

2471. *On Note* 2280.

A surface R touches a quadric Q at every point of a generator g of Q. Take g as $x=y=0$; then Q has the form

$$(x, y \between z, w) + ax^2 + bxy + cy^2 = 0.$$

The tangent plane at any point of g does not involve a, b, or c, which are parameters of the ∞^3 system of quadrics touching Q, and therefore R, at every point of g. R passes through g and the adjacent generator of Q, but it need not be ruled; for example, if it is a cubic, 2 of its 27 straight lines coincide with g.

H. P. HUDSON.

2472. *Cross-ratio of a one-one correspondence.*

If x, y are two variables in a one-one correspondence given by an equation of the form

$$axy + bx + cy + d = 0,$$

and if α, β are the self-corresponding elements, then it is well known that the cross-ratio $(x, y\,;\ \alpha, \beta)$ is constant. The purpose of this note is to prove this result and to evaluate the constant in terms of a, b, c, d. The elements α, β are the roots of

$$at^2 + (b+c)t + d = 0$$

and therefore

$$\alpha, \beta = \{-(b+c) \pm \Delta\}/2a,$$

where

$$\Delta = \sqrt{\{(b+c)^2 - 4ad\}}.$$

Thus

$$\begin{aligned}(x, y\,;\ \alpha, \beta) &= \frac{(x-\alpha)(y-\beta)}{(x-\beta)(y-\alpha)} \\ &= \frac{a(xy - \alpha y - \beta x + \alpha\beta)}{a(xy - \beta y - \alpha x + \alpha\beta)} \\ &= \frac{-bx - cy - d - a\alpha y - a\beta x + a\alpha\beta}{-bx - cy - d - a\beta y - a\alpha x + a\alpha\beta} \\ &= \frac{2(b + a\beta)x + 2(c + a\alpha)y}{2(b + a\alpha)x + 2(c + a\beta)y} \\ &= \frac{(b - c - \Delta)x + (c - b + \Delta)y}{(b - c + \Delta)x + (-b + c - \Delta)y} \\ &= \frac{(b - c - \Delta)(x - y)}{(b - c + \Delta)(x - y)} \\ &= \frac{b - c - \Delta}{b - c + \Delta}.\end{aligned}$$

If the correspondence is an involution, $b=c$, and we have the well-known result that the above cross-ratio is then equal to -1.

D. M. HALLOWES.

2473. *A lacuna and a mis-statement in the usual treatment of plane sections of the quadric.*

(i) The formulae given for the lengths of the axes of a plane section of a quadric are, in the texts available, limited to sections of central quadrics referred to the principal axes or to central sections of central quadrics referred to any set of rectangular axes through the centre. By a simple device the formulae for the lengths of any section, having a finite centre, of the general quadric can be obtained. Taking the usual notation with

$$S = \Sigma\, ax^2 + 2\Sigma\, fyz + 2\Sigma\, ux + d = 0$$

as the quadric, and $lx + my + nz + p = 0$ as the plane, we premise that there is a finite (x_0, y_0, z_0) not on $S = 0$ satisfying the relations

$$lx_0 + my_0 + nz_0 + p = 0\,,$$

$$\frac{X_0}{l} = \frac{Y_0}{m} = \frac{Z_0}{n} = \frac{W_0 - S_0}{p}\,, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

where the last equality is deducible from the previous ones, but does not replace any one of them. Transfer the origin to (x_0, y_0, z_0). The equations of the surface and plane become

$$\Sigma\, ax^2 + 2\Sigma\, fyz + 2\Sigma xX_0 + S_0 = 0\,, \qquad \Sigma\, lx = 0$$

and by (1) the curve of intersection is also that of the plane with the quadric

$$\Sigma\, ax^2 + 2\Sigma\, fyz + S_0 = 0\,,$$

and the lengths of the semi-axes may then be deduced in the usual manner to be the roots of the equation

$$\begin{vmatrix} a+\dfrac{S_0}{r^2} & h & g & l \\ h & b+\dfrac{S_0}{r^2} & f & m \\ g & f & c+\dfrac{S_0}{r^2} & n \\ l & m & n & 0 \end{vmatrix} = 0. \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

When equations (1) can be satisfied finitely and the equations

$$\Sigma\, ax^2 + 2\Sigma\, fyz = 0, \qquad \Sigma\, lx = 0$$

have coincident roots in $x : y : z$, that is, when also

$$\begin{vmatrix} a & h & g & l \\ h & b & f & m \\ g & f & c & n \\ l & m & n & 0 \end{vmatrix} = 0\,, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

(c.f., the tangential equation of a conic in homogeneous coordinates), the case of a section consisting of two distinct parallel straight lines occurs; the locus of solutions of equations (1) is also a parallel straight line and of the roots in $1/r^2$ of (2) one is zero and the other gives the inverse square of the half distance between the lines of section.

(Equation (2) can clearly be extended to n dimensions.)

(ii) The solution of the equations $\Sigma\, ax^2 + 2\Sigma\, fyz = 0$, $\Sigma\, lx = 0$ in general gives two distinct values of $x : y : z$ which are effectively the asymptotic directions

of the section. As these are independent of p it follows that parallel sections, consisting of central conics, are similar and similarly oriented. If equation (3) is satisfied, the section is parabolic or consists of parallel straight lines (which may coincide) according as

$$\begin{vmatrix} a & h & g & u & l \\ h & b & f & v & m \\ g & f & c & w & n \\ u & v & w & d & p \\ l & m & n & p & 0 \end{vmatrix} \quad \text{is not, or is, zero,}$$

and the directions of the axes of the parabolas or of the straight lines (as the case may be) are constant. If $\Sigma\, lx$ is a factor of $\Sigma\, ax^2 + 2\Sigma\, fyz$, the solution fails and excepting the three cases where the canonical forms for S are

$$ax^2 + 2wz = 0, \qquad ax^2 + by^2 = 0, \qquad ax^2 + 2ux = 0,$$

when the modifications are manifest, the finite sections consist of single straight lines, parallel to a fixed plane but not constant in direction. From these detailed considerations it is clear that a general statement of the similarity and orientation of all parallel sections requires careful qualification.

H. GWYNEDD GREEN.

2474. *A method of treating the equations*

$$\text{(i)}\ \ y = (Ax^2 + B)/x, \qquad \text{(ii)}\ \ y = (ax^2 + bx + c)/(px + q).$$

1. When A and B have the same sign, the hyperbola $y = (Ax^2 + B)/x$ has turning points and the branches lie in the acute angles between the asymptotes.

When $A > 0$, $B > 0$, $x > 0$, write

$$y = \{\surd(Ax) - \surd(B/x)\}^2 + 2\surd(AB),$$

and so y has a minimum value $2\surd(AB)$ when $x = \surd(B/A)$.

When $A > 0$, $B > 0$, $x < 0$, write

$$y = -\{\surd(-Ax) - \surd(-B/x)\}^2 - 2\surd(AB)$$

and so y has a maximum value $-2\surd(AB)$ when $x = -\surd(B/A)$.

When $A < 0$, $B < 0$, $x > 0$, write

$$y = -\{\surd(-Ax) - \surd(-B/x)\}^2 - 2\surd(AB),$$

and y has a maximum value $-2\surd(AB)$ when $x = \surd(B/A)$.

When $A < 0$, $B < 0$, $x < 0$, write

$$y = \{\surd(Ax) - \surd(B/x)\}^2 + 2\surd(AB),$$

and y has a minimum value $2\surd(AB)$ when $x = -\surd(B/A)$.

2. $y = (ax^2 + bx + c)/(px + q)$ can be written

$$y = (AX^2 + B)/X + C,$$

where

$$pX = px + q, \qquad A = a/p, \qquad B = (aq^2 - bpq + cp^2)/p^3,$$
$$C = (-2aq + bp)/p^2.$$

The branches of the hyperbola lie in the acute angles between the asymptotes if A and B have the same sign, that is, if

$$a(aq^2 - bpq + cp^2) > 0.$$

H. J. CURNOW.

2475. *Circles in contact.*

Given a chain of circles (each circle touching two others) to find the radii of the circles which touch all of them.

This problem has been dealt with from a certain angle in the case of three circles by Mr. J. M. Child in the *Mathematical Gazette*, XXXII, No. 299.

1. *Chain of three circles.*

Let r_1, r_2, r_3 be the radii of the given circles, r the radius of the required circles, and let the lines joining the centres of the given circles subtend angles θ_1, θ_2, θ_3 at the centre of the required circle.

Then

$$\cos\theta_1 = (r^2 + rr_2 + rr_3 - r_2r_3)/(r + r_2)(r + r_3), \text{ etc.,}$$

and

$$\theta_1 + \theta_2 + \theta_3 = 360^\circ,$$

which can be transformed to give

$$1 + 2\cos\theta_1\cos\theta_2\cos\theta_3 = \cos^2\theta_1 + \cos^2\theta_2 + \cos^2\theta_3.$$

Substitution gives the result

$$(rr_1r_2 + rr_2r_3 + rr_3r_1 + r_1r_2r_3)^2 = 4rr_1r_2r_3(rr_1 + rr_2 + rr_3 + r_1r_2 + r_2r_3 + r_3r_1),$$

or

$$(S_2^2 - 4S_1S_3)r^2 - 2S_2S_3r + S_3^2 = 0,$$

where

$$S_1 = r_1 + r_2 + r_3, \qquad S_2 = r_1r_2 + r_2r_3 + r_3r_1, \qquad S_3 = r_1r_2r_3.$$

The roots are

$$r = S_3/\{S_2 \pm 2\sqrt{(S_1S_3)}\}.$$

A positive root means external contact, and a negative root internal contact. If one of the given circles touches the others internally, we write its radius negative.

Examples

$r_1 = 1$, $r_2 = 2$, $r_3 = 3$; then $r = -6$ or $6/23$:

$r_1 = 1$, $r_2 = 1$, $r_3 = -(2\sqrt{3} + 3)/3$; then $r = 1$ or $(9 + 4\sqrt{3})/33$:

$r_1 = 4$, $r_2 = 4$, $r_3 = 1$; then $r = \frac{1}{3}$ or ∞ (the given circles touch a straight line in this case).

2. *Chain of four circles* (radii r_1, r_2, r_3, r_4).

Using the method of § 1, we have

$$(\cos^2\theta_1 + \cos^2\theta_2 + \cos^2\theta_3 + \cos^2\theta_4 - 2\cos\theta_1\cos\theta_2\cos\theta_3\cos\theta_4 - 2)^2$$
$$= 4(1 - \cos^2\theta_1)(1 - \cos^2\theta_2)(1 - \cos^2\theta_3)(1 - \cos^2\theta_4),$$

where

$$\cos\theta_1 = (r^2 + rr_1 + rr_2 - r_1r_2)/(r + r_1)(r + r_2), \quad \text{etc.}$$

Substitution gives the result

$$\{(r_1 - r_3)^2(r_2 - r_4)^2 - 16r_1r_2r_3r_4\}r^4 - 16(r_1 + r_2 + r_3 + r_4)r_1r_2r_3r_4r^3$$
$$- 8\{2(r_1r_3 + r_2r_4) + (r_1 + r_3)(r_2 + r_4)\}r_1r_2r_3r_4r^2 + 16(r_1r_2r_3r_4)^2 = 0.$$

When r has been calculated, the chain may be placed inside or outside the circle, according as r is negative or positive.

Examples

$r_1 = r_2 = r_3 = r_4 = 1$; then $r = -1, -1, -1 \pm \sqrt{2}$.

$r_1 = 2, r_2 = 2, r_3 = 3, r_4 = 4$; then $r = 1{\cdot}07, -7{\cdot}07, -2{\cdot}40, -2{\cdot}66$. The values which actually apply are $r = 1{\cdot}07$ and $r = -7{\cdot}07$.

When $r = 1{\cdot}07$, $\theta_1 = 81^\circ\ 18'$, $\theta_2 = 87^\circ\ 43'$, $\theta_3 = 99^\circ\ 23'$, $\theta_4 = 91^\circ\ 36'$.

When $r = -7{\cdot}07$, $\theta_1 = 46^\circ\ 28'$, $\theta_2 = 65^\circ\ 15'$, $\theta_3 = 156^\circ\ 44'$, $\theta_4 = 91^\circ\ 33'$.

H. J. CURNOW.

REVIEWS

Instrument Engineering. I. By C. S. DRAPER, W. McKAY and S. LEES. Pp. xvi, 269. 51s. 1952. (McGraw-Hill)

Although the authors, three members of the Department of Aeronautical Engineering in the Massachusetts Institute of Technology, devote page one to justifying, with the help of *Webster's Dictionary*, the title of this handsome quarto volume, their choice does less than justice to its contents. The first impression a reader might have of a book describing how to manufacture say a galvanometer or a pressure gauge is quickly dispelled by opening the book at random, or even glancing at the dust cover. It is however difficult to suggest a better name within the scope of four or five words. Briefly, the complete treatise sets out to present the general theory underlying the problem of describing and designing instruments for measurement and control. There has been a marked tendency in recent years to study such problems from a general point of view, using nomenclature and symbols which are not related to any particular technique and which are applicable equally to, say mechanical and electrical systems. Here we have what may well be the most comprehensive work on these lines yet written.

The complete treatise is in three volumes, of which only the first has been received as yet. The sub-titles of the individual volumes are as follows:

I. Methods for describing the situations of instrument engineering
II. Methods for associating mathematical solutions with common forms.
III. Applications of the instrument engineering method.

The topics covered in volume I include operating systems and components, performance operators and functions, static and dynamic operating conditions, performance equations, dimensional analysis (particular attention is paid to the use of non-dimensional quantities) and various mathematical forms for representing physical situations. Among the mathematical methods discussed in detail are those using sinusoidal response and Fourier series, while some of the ideas and methods of vector analysis are introduced in passing. This volume includes a chapter by another author giving a summary of the processes of statistics.

The method of presentation is to give in the text definitions and careful explanations of all the terms relevant to a particular topic, together with suitable symbols. These are augmented liberally by diagrammatic definition summaries, and illustrated at length in places by means of particular instruments, such as a pressure gauge and an angular velocity indicator.

The symbols used call for special mention, since they constitute one of the outstanding features of the book. They are devised, following a current transatlantic tendency, with the object of being self-explanatory. In fact they cease to be symbols in the accepted sense and have become abbreviations of the complete verbal titles of the corresponding quantities or processes, built up according to an extremely complicated set of rules. To quote two examples, the normal probability law appears as

$$\text{(NPFF)}\,\mathrm{x} = \frac{1}{\text{(SD)}\,\mathrm{x}\sqrt{2\pi}}\, e^{-\frac{1}{2}[(\mathrm{D(SR)R})\mathrm{x}]^2},$$

or again, a quantity called the airspeed meter mechanism pressure-angle sensitivity performance ratio is given by the equation,

$$\text{(PR)}_{(\mathrm{amm})[\mathrm{S}_{[\mathrm{p};\,\mathrm{A}]}]} = \frac{\mathrm{S}_{(\mathrm{amm})[\mathrm{p};\,\mathrm{A}]}}{\mathrm{S}_{(\mathrm{amm})[\mathrm{p};\,\mathrm{A}](\mathrm{ref})}}\,.$$

Compared with some symbols that appear in the book, these are very simple examples indeed. The authors do suggest rather naively that when a considerable amount of manipulation is to be carried out, complete symbols may be replaced by simpler working symbols. Nevertheless a considerable amount of hard thought on the part of the reader is necessary before the complete symbols can be interpreted, let alone recognised with facility. This being so, the whole object of the system is defeated. These remarks are not intended to belittle in any way the desirability of devising a logical set of symbols. It is however fatally easy to become a slave to such a system. Incidentally the symbols used in this book make considerable demands on the type-setter and proof reader, demands which in this case have been met very commendably.

An unusual feature for a book of this type is the use of the offset process of reproduction. The diagrams relating to particular systems are of four types: cut away, line schematic, functional block and mathematical symbol block diagrams. These are skilfully drawn and clearly reproduced. They are marred only by explanatory text that is sometimes so small that it is difficult to read.

The impression obtained after reading this volume is that it consists in the main of a very large number of definitions. It is in fact no more than an introduction to the complete work and for this reason it is impossible to offer at this stage an overall critical opinion as to its merits. Nevertheless it is clear that the task of writing it has been approached with great care and forethought. The result is a book which is certainly scholarly and which may also have considerable utilitarian value. We look forward with interest to the second and third volumes, particularly the second. B. M. Brown

Proceedings of the XIth International Congress of Philosophy. Vols. V, XIV. 1953. (North-Holland Publishing Co., Amsterdam)

These two volumes of the Proceedings of the XIth International Congress of Philosophy have been published in the new series of Monographs on Logic and the Foundations of Mathematics edited by L. E. J. Brouwer, E. W. Beth and A. Heyting. Of the very few articles which have any bearing on mathematics mention may be made of an expository paper by E. W. Beth on standard and non-standard models of formal systems (V, pp. 64–9), a paper by J. R. Myhill on some problems in recursive arithmetic (rather misleadingly called recursive function theory) and a paper by G. Kreisel on arithmetical models for consistent formulae of the extended predicate calcalus (XIV, pp. 50–9 and pp. 39–49 respectively). R. L. G.

Truth and Consequence in Mediaeval Logic. By E. A. Moody. Pp. viii, 113. 1953. (North-Holland Publishing Co., Amsterdam)

To one brought up to think of the fourteenth century as belonging to the dark ages it is both refreshing and exciting to find how close the minds of mediaeval logicians like William of Occam, Albert of Saxony and Jean Buridan approach to contemporary thinkers. Discussing the paradox of sentences which reflect on themselves, like " the sentence of page one of Antinome is false ", where in fact that is the sentence on page one of the book called *Antinome*, Occam maintains that a proposition cannot say something about itself since a sentence cannot be part of itself, which is exactly the view which we find in Wittgenstein's *Tractatus logico-philosophicus*. Buridan finds this solution inadequate, showing that the paradox may be expressed in a form in which the self-reference is indirect, and proposes a solution based on denying the equivalence of " p " and " p is true ", a denial which has been reiterated in some current writings on logic, though for reasons rather different from Buridan's.

Moody undertakes the task of translating mediaeval logic into a modern symbolism—not the symbolism of *Principia Mathematica* but one more akin to the systems of Lewis and Langford, introducing the idea of possibility. The sources of the formalised definitions and laws are given in footnotes so that the reader may test for himself the faithfulness of the rendering ; the majority of the formalisations are in fact remarkably apt, but in one or two places the formula does not appear to cover the rule as closely as one would wish. For instance, formula 2.11 (on p. 89),

$$(p \cdot - q) : \dashv \cdot - (p \dashv q),$$

which says that it follows from p and not-q that q does not follow from p, for *impossibile est ex vero sequi falsum.*

The formalisation introduces five axioms which are not explicitly stated in mediaeval writings, namely,

$$pq \dashv qp, \quad p \dashv pp, \quad (pq)r \dashv p\,(qr),$$

an existence axiom asserting that there are propositions p and q such that

$$-(p \dashv q) \,.\, -(p \dashv -q),$$

and the modal axiom,

$$\Diamond p \dashv - \Diamond(-\Diamond p),$$

where $\Diamond p$ says that p is possible. Buridan's definition of "p is true" is expressed by the formula

$$(Ex)\ `p_s'\ x.\ (x=p)$$

(there is an x such that 'p' stands for x and $x=p$).

R. L. Goodstein.

Undecidable Theories. By A. Tarski, A. Mostowski and R. M. Robinson. Pp. 98. 18s. 1953. Studies in Logic and the Foundations of Mathematics. (North Holland Co., Amsterdam)

The decision problem for a branch of logic or mathematics is the problem of devising a mechanical test for the truth or falsity of all its statements. In sentence logic, for instance, such a test is provided by the truth tables, for granted sufficient time any compound sentence formed by combining sentence variables may be tested by the tables without recourse to imagination, discovery or invention. In the same way the sieve of Eratosthenes provides a decision procedure for all statements of the form "n is a prime". A decision procedure to be acceptable must be in some sense finitist, and the testing of an infinity of cases (as in some familiar arguments in mathematics) is not allowed. Of course a branch of mathematics, for which the decision problem has been solved affirmatively, contains no insoluble problem. The most interesting parts of mathematics are generally speaking those which do not admit a decision procedure, and in this book the authors are concerned almost entirely with proving that no decision procedure exists for certain well-known branches of mathematics. Amongst the general results proved it is shown that there can be no solution to the decision problem of the second degree, that is to say, there does not exist a mechanical procedure for deciding of any axiomatic system whether it is decidable or not, so that the decision problem itself constitutes one of the "interesting" branches of logic.

Historically the first proof of undecidability was given by Alonzo Church (based on the work of Kurt Gödel) who proved in 1936 that the restricted predicate calculus does not admit a decision procedure. Church identified the problem of finding a decision procedure for a system of statements

$S_1, S_2, \ldots$ with that of finding a *computable* function $f(n)$ such that, for each n, $f(n)=0$ if, and only if, S_n is true.

Undecidable systems, that is, systems which can be *proved* to have no decision procedure are not necessarily highly developed systems, as the following example shows.

We consider a system in which the statements are all of one of the forms $f(x)=0$ or $f(x)=1$, where f is a computable function and every expression of one of these forms is a statement, and we suppose that the elements, statements and proofs of the system are numbered off, and that there is a computable function $s(t)$ such that, for any value of t, $s(t)$ is the number of the sentence which results from substituting t for the first variable in the sentence numbered t. If this system is decidable there is a computable function $m(x)$ such that $m(p)=0$ if and only if p is the number of a true statement, and $m(p)=1$ otherwise. Let $\mathbf{t}$ be the number of the statement " $m(s(x))=1$ ", so that the number of the statement " $m(s(\mathbf{t}))=1$ " is $s(\mathbf{t})$. If the statement " $m(s(\mathbf{t}))=1$ " is true, then since its number is $s(\mathbf{t})$, $m(s(\mathbf{t}))=0$, which contradicts our hypothesis; on the other hand if the statement " $m(s(\mathbf{t}))=1$ " is false, then $m(s(\mathbf{t}))=1$ is true, so that either hypothesis leads to a contradiction and therefore the existence of the decision function $m(x)$ is impossible.

Undecidable theories is a book formed by bringing together three essays, the second of which, on the undecidability of arithmetic, is the work of all three authors, and is the product of certain simplifications which Robinson effected in some earlier work of Tarski and Mostowski. The first essay, on a general method in proofs of undecidability, and the third, on the undecidability of the elementary theory of groups, are by Tarski alone. The essays have been well edited and have been carefully knit together, but they do not by any means form a self-contained whole and it is a great pity that the attempt was not made to make the book more independent of other recent publications. A considerable part of the second essay stands on its own feet, apart from appeals to the theory of general recursive functions, but, towards the end, the essay relies on results which Tarski obtained in his construction of a decision procedure for elementary algebra (without variables for natural numbers), and a result from Ryll-Nardzewski's paper in *Fundamenta Mathematicae* on the role of the axiom of induction (1952). The first essay, however, is difficult to read without some previous familiarity with the subject, including for instance some results of Janiczak's in the *Journal of Symbolic Logic* (1950), Herbrand's deduction theorem and some early papers of Tarski's.

The key theorem used in the proof of the undecidability of arithmetic is remarkably simple and powerful. Suppose that we have a certain (1, 1) correspondence between the natural numbers and the expressions of some formal system T and let E_n be the expression correlated with the number n, and $N(\phi)$ the number correlated with the expression ϕ. Further let D_n be the number correlated with the expression obtained by substituting the nth numeral for the first variable in the expression whose number is n, and let V be the set of all natural numbers n such that E_n is a sentence valid in T. Then, if the theory T is consistent, *the function D and the set V are not both definable in T*. The proof of this result is along the lines described above and is a metamathematical reconstruction and generalisation of the argument involved in the famous antinomy of the liar. An immediate consequence is that if all recursive functions are definable in a consistent system T then T (and every consistent extension) is undecidable. On the other hand, every function definable in a system T with a recursive system of axioms is shown to be recursive, and from this it follows that a consistent system is undecidable if it admits a non-recursive function or set. The next step is to construct a number of subsystems of arithmetic each of which (before the last) is shown to

contain the next ; the last of these is shown to admit all recursive functions so that both the subsystem itself, and every consistent extension, are undecidable. One of these extensions is finitely axiomatisable (that is, its axioms are finite in number and are formulated without predicate variables) and so provides an example of a finitely axiomatisable non-decidable system (simpler than the example obtained by Mostowski and Tarski in 1939).

The third essay establishes the undecidability of the elementary theory of groups. The system considered, called G, is elementary in the sense that its logical basis is the restricted predicate calculus ; the only non-logical axioms of the system are

$$x.(y\,.\,z)=(x\,.\,y)\,.\,z,\quad (E\,x)\,(x=y\,.\,z),\quad (E\,y)\,(x=y\,.\,z).$$

The proof proceeds by extending the undecidability of arithmetic to a theory of integers. This theory of integers is then modified by a relativisation of quantifiers, and the modified system, J^*, is shown to be weakly interpretable in (an inessential extension of) G, from which the undecidability follows of G. Though undecidable, G is *not essentially undecidable*, for G may be extended by adding the commutative law to the axioms to form the elementary theory of Abelian groups which has been proved to be decidable by Tarski's pupil Wanda Szmielew. Amongst sub-theories of G which are undecidable are the elementary theory of groupoids and the elementary theory of semi-groups.

R. L. Goodstein.

Statistics for Technologists. By C. G. Paradine and B. H. P. Rivett. Pp. vii, 288. 25s. 1953. (English Universities Press)

This book is written for engineers and practical scientists who need to learn the basic ideas of statistics and probability theory and who have reached a standard of mathematics higher than that which most introductory statistics books demand from their readers. The authors combine experience of teaching mathematics with practical experience of the application of statistics.

The plan of the book is conventional, but because of the higher knowledge of mathematics expected of its intended readers its subject matter ranges further and goes deeper than is usual in introductory books. It begins with a discussion of frequency distributions and the basic probability distributions. The uses of the χ^2-test for testing goodness of fit, of the t-distribution for testing the significance of the difference of means of small samples, and of the F-test for comparing variances are then described. There follow two chapters on the applications of the theory, first to the Quality Control of mass produced components, and secondly to Sampling Inspection Schemes, the chapter on which includes a short introduction to sequential sampling. After two chapters on the Method of Least Squares and Correlation the book concludes with a chapter on Analysis of Variance and a rather unexpected but helpful introductory chapter on the Principle of Maximum Likelihood and Probit Analysis. The necessary statistical tables are included.

The text is illustrated throughout with well-chosen practical problems ; useful collections of exercises (with answers) are included. The authors' style is terse but their explanations are clear and adequate.

The book will be useful to practising engineers and scientists who need a first book on statistical methods ; if they need more advanced techniques than the book describes they will find that references are given to the appropriate specialist works. It will also meet the need for a sound text-book appropriate for many of the basic courses in statistics now given to students of the applied sciences in universities and technical institutions.

B. C. Brookes.

Mechanik der Festkörper. By ERWIN LOHR. Pp. viii, 483. DM. 39.60. 1952. (Walter de Gruyter & Co., Berlin)

Professor Erwin Lohr died before he had completed the three volumes of the textbook of physics which he had planned. The first volume of the work was written before his death and is presented by the publisher in the above form. Perhaps a title more likely to suggest general theoretical mechanics than the present one would have been more appropriate, since it is not until p. 206 that the mechanics of deformable bodies is treated. This is understandable if the author's intention was to present a comprehensive treatise on mathematical physics, but it is misleading to have a single volume carrying such a title when only just over half of its contents would conventionally be so described.

The first half of the book contains ten chapters which are concerned mainly with the mechanics of a particle, systems of particles and rigid bodies. It begins with a general account of the properties of space and time and of the co-ordinate systems used to measure both. This is followed by a chapter on vector analysis in which are established such properties of vectors as are required throughout the rest of the volume. The study of mechanics proper begins with an account of the laws of motion and the formulation of the equations of dynamics of a system of bodies and of a continuum in vector form.

There is then a short digression on the theory of the gravitational field, after which the author returns to the theory of the motion of rigid bodies. A discussion of the principles of the conservation of energy and of momentum is given, though very few examples are given to illustrate them.

The dynamics of rigid bodies is then considered in a little more detail. Euler's equations are derived and applied to the discussion of the motion of a top. Rather surprisingly, there is here interpolated a section on the spherical pendulum after which the author develops the theory of statics and of machines (with and without friction).

The general principles of dynamics are then discussed starting from d'Alembert's principle and the principle of virtual work. The idea of generalised co-ordinates is introduced, Lagrange's equations derived, and the relations of these equations to Hamilton's principle shown. Hamilton's cononical equations are established and the dynamical theory due to Hamilton and Jacobi developed briefly. In the chapter which follows planetary motion, Foucault's pendulum and the motion of projectiles are used to illustrate the principles previously established.

The second half of the book is concerned with the mechanics of deformable solids. The first chapter contains an account of the analysis of stress and strain in a solid body and of the stress-strain relation in the case of an elastic solid ; the fundamental relations of the theory of elasticity are then derived. In the next chapter relations are found concerning the strain energy function, and the theorems of Castigliano and St. Venant are discussed. The next chapter is devoted to applications of the general theory developed in the two previous chapters to problems of torsion, compression and bending. This is followed by a chapter on the physics of solids which discusses the form of the stress-strain curve for various types of material, the physics of plasticity and the meaning of such concepts as hardness. The sections on plasticity are much fuller than is normal in a work of this kind.

The remainder of the book deals with various types of dynamical problem associated with deformable solids. The fifth chapter of the second part begins with an account of the theory of elastic vibrations of systems with one or two degrees of freedom, such as the damped and forced oscillations of a spring, and the coupled oscillations of two springs. To deal with problems of this kind the theory of Fourier series and of the Laplace transform is given briefly in the text. Chapter VI outlines the theory of the one-dimensional wave-equation

with special reference to the transverse vibrations of an elastic string, and the next chapter extends the analysis to cover the vibrations of two- and three-dimensional elastic systems (such as membranes and plates). The book ends with a chapter on impulsive motion, much of which might better have been included in the first part.

It will be seen that the author covers a wide variety of topics and at times the reader cannot help but feel that the whole work would have been considerably improved if the material it presents had been better organised. Another defect of the book is that it uses the Gibbs notation for vectors and tensors and though this is suitable for the discussion of the dynamics of rigid bodies it is exceedingly cumbersome in the theory of elasticity. The individual sections are well written, however, and anyone responsible for the teaching of applied mathematics at an advanced level will find much of value in some of them, but, because of the existence in English of several admirable works covering the same range of topics (but in more than one volume!) he is unlikely to recommend this particular work to the attention of his students. I. N. SNEDDON.

Finite Deformation of an Elastic Solid. By F. D. MURNAGHAN. Pp. 140. 32s. 1951. (Chapman and Hall, London)

The classical theory of elasticity, as embodied in Love's *The Mathematical Theory of Elasticity*, is based on the assumption that, in the problems considered, the deformation of the solid bodies is so small that the squares of the strain components are negligible. In the last twenty years various attempts have been made to develop a more general theory of *large* elastic deformations, some of them based on the fundamental physical properties of a particular substance, such as rubber, and others independent of the form of the fundamental stress-strain relationships of the material. Among investigations of this latter kind that of Murnaghan in 1937 is of great importance, so that a book on the subject by the same author is bound to be greeted with interest.

The present book, which is based on lectures given in Europe and both the Americas, aims at the presentation of a unified treatment of the influence of squares and higher powers of the strain components in the theory of elasticity.

It begins with a chapter entitled " Vectors and Matrices " whose contents will be familiar to any student of applied mathematics who is otherwise equipped to begin the study of elasticity. Some of the author's terms will baffle even the pure mathematician already acquainted with the theory of matrices ; for example, the matrix A is said to be " polite in multiplication if AB is the same as B for every B ". Meeting such a phrase early in the book forces the reader to work through accounts of a subject he already knows, if only to ensure that he is using words in precisely the same sense as the author. That is not, however, a serious complaint. The real defect of this chapter is not that it is written in an unorthodox way, but that for the purpose undertaken it is quite unnecessary. Murnaghan, in his 1937 paper (*Amer. Journ. Math.* 59, 235) showed that in the analysis of stress problems the use of tensor calculus is both effective and elegant. In the present treatment it seems at first sight as though the use of matrix methods is simpler (though in places the going is rather heavy!) but the author reverts to tensors in at least one place, and when he comes to the discussion of problems in cylindrical and spherical polar co-ordinates he has to quote, without proof, results which would have followed naturally from a tensor treatment of the fundamental equations. Since a complete account of tensor calculus exists in Sokolnikoff's companion volume in the same series, Murnaghan could have begun his treatment of elasticity by presupposing a knowledge of the contents of some at least of the chapters of such a book.

The second and third chapters contain an account of the specification of strain and the connection between stress and strain. In the fourth chapter the author applies these general results to the consideration of an isotropic elastic medium and develops his " integrated linear theory " of hydrostatic pressure which attempts to explain certain of Bridgman's experiments on the compressibilities of various media up to pressures of 105 atmospheres. The author's basic assumption that the Lamé constants are linear functions of the pressure does not seem to be in accord with the general theory of finite strain, so it would be unwise to regard this section as more than a semi-empirical approach to the problem raised.

After a chapter on non-isotropic media the book is devoted to the solution of special problems. These last two chapters are perhaps the most interesting in the book, containing as they do results not already published elsewhere, such as the special forms assumed by the cubic terms in the strain energy for aeolotropic materials, second order effects in shearing stress and second order effects in the torsion of a cylinder. In the discussion of these problems no reference is made to the general solutions valid for arbitrary strain energy derived in recent years by Rivlin and A. E. Green.

From these comments it will be seen that the present book has not been entirely successful. Its failure may, in part, have been due to an uncertainty in the author's mind as to the public for which he was writing. If he had formulated his book in terms of tensor calculus it might not have reached such a wide public, but it would have been of greater value, and would have had a greater effect on the development of a growing subject. (The student who is not prepared to learn tensor calculus will probably find it too difficult in any case.) Professor Murnaghan could not do otherwise than write an interesting book, but, in this instance, he has, by restricting himself to his own researches, lost the opportunity of writing a truly authoritative account of the subject. He makes no reference to the great advances which have been made in this field by Rivlin, Green, Signorini and Reiner—indeed the only references in the whole book are to the author's own elementary textbooks ! In a subject that is growing rapidly in importance in engineering and applied physics the time is ripe for a comprehensive treatment of the whole field of research. The present book will certainly have to be taken into account when such a treatment is written, but it is not the work we had hoped for from one of the masters of the subject. I. N. SNEDDON.

Kernel Functions and Elliptic Differential Equations in Mathematical Physics. By S. BERGMAN and M. SCHIFFER. Pp. xiii, 432. $8. 1953. (Academic Press. New York).

We may hope that von Mises saw this book, dedicated to him, before his death a few months ago ; for in spirit as well as substance it forms a sequel to the relevant sections of Frank and von Mises, *Die Differential- und Integralgleichungen der Mechanik und Physik* (2nd edition, New York, 1943).

The essence of the book is the study of the solution of a partial differential equation as a functional of the domain, the boundary values and the coefficients of the equation. For instance Dirichlet's problem is to find a potential function with given boundary values, Neumann's problem specifies a given normal derivative on the boundary ; basic solutions are provided by Green's and Neumann's functions, while Robin's functions are appropriate to a more general problem. Such " kernels " play a central part in this volume. In fact, pp. 258–403 give a detailed study of the theory of the equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=q(x, y)\,u(x, y)$$

in terms of the fundamental functions. To plunge at once into such an abstract problem would probably be too severe an exercise for the average engineer or physicist, and so the authors have wisely chosen to prepare the way by a long introductory section (pp. 1–257) in which the principles of heat conduction, fluid dynamics, electrostatics and elasticity are related to the governing partial differential equations and their boundary conditions, and the various fundamental solutions are exhibited without too much insistence on mathematical rigour, and their physical significance displayed. The reader should have some elementary knowledge of these topics in applied mathematics, as well as a very firm grasp of partial differentiation. But thus equipped he will begin to see the common mathematical core of these doctrines, the importance of the various kernels, and perhaps even the need for a logical and more abstract discussion such as the second part of the volume offers. Even so, the second half demands much closer attention and concentration than the less closely woven and more physical first half. In the preface, we read that the student is assumed to have " a fair acquaintance with the standard methods of analysis provided, for example, by the excellent books of Courant-Hilbert, Frank-Mises, and Jeffreys. For this reason, an engineer or a physicist with the conventional mathematical training may possibly find some parts difficult to read ". This is certainly true of the later chapters, where the authors frequently endeavour to summarise work recently published in research papers ; the content is valuable, but the exposition is not as smooth as in the earlier chapters.

Frequent references for further reading are given, to books and memoirs. Supplements to the physical background or developments and expansions of the mathematical arguments may have to be sought by the tyro, who should find little difficulty in locating what he needs. The typography and lay-out are both competent and pleasing. T. A. A. B.

Mécanique Générale. By J. Pérès. Pp. iv, 407. 2100 fr. ; bound, 2545 fr. 1953 (Masson, Paris).

The uniform excellence of French treatises on analytical dynamics is traditional, going back at least as far as Lagrange ; at the present day, there are many topics on which no author need be ashamed of copying the master-strokes of Appell. There is still room, however, for experiment in the presentation of principles, in the light of the needs of particular classes of students. Professor Pérès expects his readers to have had a sound first course in dynamics ; for them he provides a bridge to the domain of advanced specialised studies. After a discussion of Newton's laws and applications to rigid bodies, he makes what is practically a fresh start with D'Alembert's principle and the method of virtual work, as a preparation for the variational principles of later work ; particular attention is paid to the nature and effect of constraints. A special chapter is devoted to a study of the equation

$$\dot{q}^2 = F(q)$$

which occurs in so many places in dynamics, and then the general methods are applied to the motion of a particle, free or constrained, and the motion of spheres and tops. Now follow the equations of Lagrange, Appell and Hamilton, and the elements of principles of variation, and a chapter on stability of equilibrium and steady motion. The chapter on impulses again gives opportunity for a close examination of constraints, with praiseworthy emphasis on the geometry of the topic. A long final chapter on deformable bodies is chiefly concerned with the statics and dynamics of flexible strings, and does not pretend to be more than an introduction to the general problems of this field.

Although much of the material is classical, there are many points of novelty and interest in the exposition; the close examination of some details often dismissed very hastily in our own text-books can be recommended to those who lecture on dynamics in our universities, while it may be added that they will not find a single rod of mass m and length $2a$. Where precisely this volume fits into the great French hierarchy of such books it may presumptuous to discuss; naturally it does not pretend to rival the massive comprehensiveness of Appell, but it digs more deeply into principles than de la Vallée Poussin's brilliant and readable two volumes of *Mécanique analytique*.

I have always admired the printing and format of Masson's publications, and this volume is well up to their high standard. T. A. A. B.

Contributions to the Theory of Non-linear Oscillations. II. Edited by S. Lefschetz. Pp. v, 116. 10s. 1952. *Annals of Mathematics* Studies, 29. (Princeton University Press; Geoffrey Cumberlege, London)

This sequel to an earlier volume (for a review, see *Gazette*, XXXV, p. 282, December 1951) contains six research papers on differential equations. It is clearly not intended to be a systematic survey of recent progress, but it covers a wide range of topics and is therefore a valuable addition to the specialist literature on non-linear oscillations.

On the analytical side, there are papers by: (1) M. L. Cartwright, on approximations to the period and amplitude of solutions of van der Pol's equation

$$\ddot{x} + k(x^2 - 1)\dot{x} + x = 0,$$

for large k; (2) E. A. Coddington and N. Levinson, who examine periodic solutions of an "almost linear" system of equations

$$\dot{x} = Ax + kf(x, t, k)$$

for small k by perturbation methods; (3) J. McCarthy, on the calculation of limit cycles by successive approximations; (4) H. L. Turrittin, on asymptotic expansions for solutions of linear differential equations containing a large parameter. On the topological side, a paper by H. F. De Baggis discusses what topological features of the trajectories of a system

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y)$$

are unaffected by small changes in P and Q; two notes by S. Lefschetz describe the behaviour of these trajectories near a critical point (where $P = Q = 0$), and their behaviour in the large for the system

$$\dot{x} = y - k(\tfrac{1}{3}x^3 - x), \quad \dot{y} = -x$$

(equivalent to van der Pol's equation). G. E. H. R.

Traité de Physique Théorique et de Physique Mathématique. I. Methodologie, Notions Géométriques. By J. L. Destouches. Pp. xiv, 228. 3000 fr. **III. Éléments de Théorie des Quanta et de Mécanique Ondulatoire.** By L. de Broglie. Pp. vii, 302. 3000 fr. 1953 (Gauthier-Villars, Paris).

M. Destouches is the editor of this new *Traité* and is the author of the first two volumes (the second, on Newtonian mechanics, being not yet published). While the series is intended to form a general survey of modern physical theories, one learns from his general introduction, and from M. de Broglie's

preface to his own volume, that its particular structure is determined to a great extent by its close association with the courses given in the Faculty of Sciences in Paris. Part of its object appears to be to encourage and facilitate the provision of similar courses elsewhere in France. The differences in methods of teaching and study between French and British universities, and, to some extent, the differences in intellectual view-points, make the systematic use of the series as a whole rather unlikely in this country. Nevertheless, it will be referred to with interest, particularly for the sake of seeing possible ways of approach to the presentation of fundamental concepts.

In the first section of his book, on methodological preliminaries, M. Destouches seeks to lay very deep foundations. He gives a philosophical discussion of notions such as non-contradiction and consistence in deductive theories, inductive synthesis, the principle of " positivity " in physics, and so on. Such an introduction would scarcely prove congenial to the minds of most physics students in Britain. But, with its copious references and bibliography, it deserves the attention of students of the philosophy of science.

The second section, on geometrical notions, contains chapters on objects, sets and spaces, on projective and affine geometry, on metrical euclidean geometry, and on the geometry of masses (mass-centre, tensor and ellipsoid of inertia, etc.). The treatment is highly abstract, more so, in fact, than that adopted in this country in most courses of *pure* mathematics dealing nominally with the same topics.

Readers should note the extensive table of symbols at the end of the book; it is evidently intended for continual use along with the text.

M. Louis de Broglie also devotes much attention to preparing the ground. He has five chapters on the theories of Maxwell and Lorentz, special relativity, statistical mechanics, and the theory of equilibrium radiation. " Quantum " notions first appear in his chapter on photons. The next four chapters lead up to the ideas of wave mechanics following, in the main, the historical order of development. Quantization in wave mechanics is then dealt with. This is illustrated by a few standard examples, briefly discussed, but the book is largely concerned with generalities; it appears that another author will deal with more detailed applications. The remaining chapters are on the quantum mechanics of Born, Heisenberg and Jordan, the probability-interpretation of wave mechanics, electron spin and Dirac's theory, Pauli's exclusion principle, and quantum statistics.

The gap is large that remains between such an account and applications to, say, the theory of spectra or the theory of collision phenomena and even more, of course, between it and nuclear theory, quantum electrodynamics and modern field-theories. It makes one question whether a treatment at such length—300 pages—can be accommodated in any routine course of physics. At any rate probably no one is better qualified than M. de Broglie to write upon this part of the subject. W. H. McCREA.

The Stability of Rotating Liquid Masses. By R. A. LYTTELTON. Pp. vii, 150. 35s. 1953. (Cambridge University Press)

This is a clear, well-written and single connected account of the classical stability problem for rotating incompressible liquids. It has been of great interest to the reviewer for three reasons. In the first place and probably of greatest importance, the book corrects certain statements and conclusions due to Jeans concerning the cosmogonical application of the theory and the author arrives at the conclusion " that the dynamical evidence is entirely adverse to the so-called fission process of formation of binary systems." In the second place the book acquires interest because of its clear exposition of the concept of the " exchange of stabilities ", that is the evolution of the stability of a

dynamical system whose co-ordinates are functions of a single parameter. In the third place the book deals with the theory of Ellipsoidal Harmonics and Lamé Functions in such considerable detail and so fluently that these chapters are probably the best introduction to this branch of Function Theory.

Concerning the application of the theory to stellar evolution it must be stated that the classical assumptions are themselves open to criticism. The author points out that the linearisation of the equations of motion is in itself questionable but it must be borne in mind also that the presence of a temperature field or a velocity shear within the rotating gas can very seriously modify the conclusions of the classical theory. T. V. DAVIES.

A School Course in Mechanics. II. By A. J. BULL. Pp. viii, 157–408. 15s. 1953. (Cambridge University Press)

This volume contains a complete Sixth Form course in Mechanics for all except mathematical specialists, and for them it will provide fully two years' work. The subject matter is arranged in logical sequence and is dealt with most carefully and thoroughly, the explanations of the general principles being excellent. The work on applications to particular problems is very well done and is assisted by an admirable collection of diagrams.

A summary of the contents follows, with critical notes, where necessary, on each chapter.

XIX. Non-uniform motion in a straight line. This includes some difficult work, but is an excellent treatment of the subject and leads on to the next chapter. XX. Simple harmonic motion. The treatment is restricted to motion in a straight line. The author fails to consider the most general initial conditions and does not derive the fundamental equation

$$x = x_0 \cos nt + \frac{u_0}{n} \sin nt.$$

XXI. Two-dimensional kinematics. Radius of curvature is not mentioned, only circular motion being considered. XXII. Motion of a particle in a plane (general principles). Here are given very clear explanations of the formation of the equations of motion (the author insists on two diagrams, one for forces and one for accelerations) and the use of the principles of energy and linear momentum. XXIII. Motion of a particle in a plane (applications). Projectiles and circular motion are dealt with, and the work is very well graded. There is not enough work on the enveloping parabola to satisfy mathematical specialists. XXIV. Application of Newton's Laws to work with a moving origin. XXV. Laws of conservation. The method of differentiating the energy equation is used, and there is a good collection of examples on the use of the energy and momentum equations. There is a good discussion on impact, but not enough easy examples on direct impact. Some work is done on the motion of two connected particles in a plane, and on the motion of the centre of gravity of a system of particles. There follows a set of test exercises on the dynamics of a particle, questions from various H.S.C. examinations being arranged according to topics. This should prove very useful to schools. XXVI. Motion of a rigid body about a fixed axis. The chapter includes calculations of moments of inertia by integration, general theorems on moments of inertia, the equation of angular acceleration, the use of the energy principle, reactions, impulses, and the principle of conservation of angular momentum. XXVII. Motion of a rigid body in a plane. Here the treatment is rather cursory, though there is a good discussion of energy. On p. 305, in the paragraph headed " Use of the instantaneous centre ", it seems to be assumed that the equation $N = I\ddot{\theta}$ can always be used in such cases; some modification of this statement seems to be called for. Further test exercises

on rigid dynamics follow. XXVIII. Coplanar systems of forces. An excellent chapter dealing with reduction of forces in a plane, force and funicular polygons, conditions of equilibrium, jointed bodies, three-force problems, light frames (including graphical methods), bending moments. Further test exercises on statics follow. XXIX. Miscellaneous methods (virtual work, potential energy, stability, small oscillations). The explanations here are rather short, but the worked examples make the methods clear. XXX. Hydrostatics. (Pressure in a fluid, total thrust, centre of pressure, floating bodies.) Adequate treatment with plenty of integration.

The author and the Press are to be congratulated on an excellent piece of work. F. J. TONGUE.

Advanced Mathematics in Physics and Engineering. By A. BRONWELL. Pp. xvi, 475. \$6.00 (51s.). 1953. (McGraw-Hill)

The scope of this book differs from that usually to be expected under the heading of Mathematics for Technical Students by developing a few " fundamental mathematical formulations " to " provide a broad perspective of the physical sciences ". The author aims to emphasise the unity of the mathematical treatment of those sciences, and provides a double fugue on the subjects of the solution of differential equations and vector analysis, with a coda based on the use of functions of a complex variable. The exposition of each subject is followed by several chapters, each applying the results obtained to a particular subject. In general, the treatment is clear, the development is in a logical sequence, and the degree of mathematical rigour is adequate for the intended readers ; in any case, there are well chosen references to sources of information on the details omitted. However, the author has a habit of quoting much more exact definitions of mathematical concepts than will be understood from, or are needed by their context. Another source of obscurity in the presentation is the occasional omission of the vital sentence, as for instance where a section or characteristic lines in supersonic flow fails to state what the lines are, while leaving a reasonably high probability that they can be identified with Mach lines. Again, one must point out that several terms such as regular, linear, linearly independent are used several chapters before any explanation or definition of their meaning is given.

As a prelude, three chapters cover infinite series, complex numbers (from $i=\sqrt{(-1)}$ as a start), and Fourier series and integrals. A good " user's " summary of convergence leaves the impression that uniform convergence is a quibble, and p. 1 is marred by the following method of testing the convergence of a series of terms involving a complex *variable*. " By separating real and imaginary parts of a complex series and equating reals and imaginaries on both sides of the equation, it is often possible to reduce a single complex series to two real series. . . . Each series can then be tested for convergence."

The exposition of the theme of differential equations covers ordinary differential equations, series solutions, Bessel and Legendre functions and partial derivatives. It seems a pity that the Laplace transform is omitted here, and kept to the last chapter of the book. The applications are mainly to oscillations, in electrical and elastic problems both with lumped and with distributed elements, and to Lagrange's equations. The vector analysis then includes operations and integral theorems, and is followed by a countersubject, the wave equation : they are applied in chapters on heat flow, hydrodynamics and electromagnetic theory. The final chapters, on functions of a complex variable, include applications to two dimensional field problems, and to dynamical stability criteria. It will be seen that the level of the treatment means that the emphasis in the title is on the physics rather than the engineering.

There is a good supply of problems in most of the chapters, and errors seem to be few. Was it a matter of chance that the answer to the example checked by the reviewer (p. 285 No. 33) is wrong? Other errata are: p. 37 missing l in equation (1); p. 59, missing Φ in (8); p. 28, n for m in 1.17; p. 390, $n=2m$, in the line before (7); p. 79, Fig. 3, the intersections of the dotted curve are inaccurate; p. 36, 1.7 " two sets of function $\Phi_m(x)$ and $\Phi_n(x)$ " should read " two functions $\Phi_m(x)$, $\Phi_n(x)$ of a set ". R. B. H.

The Classical Theory of Fields. By L. LANDAU and E. LIFSHITZ. Translated from the Russian edition of 1948 by M. Hamermesh. Pp. ix, 354. $7.50. (Addison-Wesley Press, Cambridge, Mass.)

The word " classical " in the title is intended to show that all quantum phenomena are omitted from this book. But within that context it admirably fulfils its aim of providing " a systematic presentation in the theory of electromagnetic and gravitational fields ". That is to say, it is clear, it is inclusive, with sections on geometrical and wave optics, it is powerful and it is original. There is hardly anyone who will not find something novel in the treatment here provided. And the liberal supply of problems which intersperse the text are a most valuable asset. Right up to the last two chapters the work is essentially relativistic in the special sense: only at the end is general relativity introduced.

Some people may wonder why there is no mention of continuous media. This is partly due to the restriction to classical physics, in the usage of those words previously mentioned; it is apparently also due to the general plan of the course in theoretical physics for which the authors are responsible, and of which this book is simply one part. But within these self-appointed boundaries there is a really first-rate account of the electrodynamics of the vacuum and of point charges. The book, despite its mathematical elegance, is almost wholly self-contained. The necessary tensor analysis is developed as the book goes along.

But there is something more to be said. In their preface the authors say that " as a starting-point for the derivation of basic relations, we use the variational principle, which enables the attainment of maximum generality together with an essential simplicity of presentation ". This is quite right; and it confers an austere beauty upon the whole account. But at the same time it does violence to every physical insight (except the last) which has led to the development of the complete picture. We may say that the authors " derive " everything from the Principle of Least Action. But what is the basis on which this principle rests?—on almost a whole century of inductive effort, starting with the inverse square law of force between two charges, and ending with the genius of Maxwell's displacement current. And what do we find in this account?—the inverse square law is " derived " on page 94, and, as Professor Rosenfeld has pointed out, Maxwell's equations are themselves " derived " on page 66 from certain additional terms in the Lagrangian function, whose only ground for existence is that " it is found to have the form . . .". The same magic words " it is found " appear on page 42 to describe the interaction of a field with a charged particle.

All this means that we have made a mathematical book out of a series of somewhat disjointed physical intuitions: we have a deductive account of a subject which grew up inductively. Perhaps this is the chief difference between theoretical physics and applied mathematics! But whether or not this be the case, it will be the mathematicians who are most at home in this book; only they and the physicist who " knows it all first " will be able to enjoy the undoubted merits of this account. Despite its simplicity it is a book for the advanced student—and a good book too. C. A. COULSON.

Introduction to Elliptic Functions. By F. Bowman. Pp. 115. 12s. 6d. 1953. (English Universities Press)

This little book is intended for the physicist or engineer. It is therefore not concerned with the general theory of elliptic functions but solely with the Jacobian functions and elliptic integrals. The need for them in physics and engineering arises largely in relation to the use of the Schwartz-Christoffel transformation to solve problems of flow in two dimensions. This transformation is considered in some detail in Chap. VI but it is assumed that the reader has some knowledge of the complex variable and is familiar with the idea of conformal representation.

The development by stages of the pure mathematics and applications is skilfully achieved. In Chap. I it is first shown how the properties of the circular functions could be based on the definition of the inverse sine as an integral. Jacobi's functions are analogously defined and the fundamental identities, formulae of differentiation and periodicity for a real argument established. In the next chapter elliptic integrals are considered but examples are restricted to those involving the first and second kinds without preliminary reduction. There is a short section on the use of tables—as given by Dale, Jahnke and Emde, and Milne-Thomson. Applications to arcs, surface areas and some problems in dynamics follow in Chap. II. The definitions are next extended to allow of the complex argument, involving double periodicity. Conformal representation, using elliptic functions, is treated in Chaps. V and VI and applied to problems of electricity and hydrodynamics in Chap. VII. Landen's transformation is introduced by means of the solutions of two such problems whose equivalance is intuitively obvious from physical considerations. A further chapter on conformal representation deals with the elliptic integral of the third kind. Methods of reducing $\int dx/\sqrt{X}$, where X is a quartic or cubic in x, to standard form are given in Chap. IX. In the last chapter, X is a particular kind of quintic ; after a formidable series of transformations it is shown that the problem has been solved of finding the capacity of a condenser whose cross-section is in the form of two concentric, similarly situated, squares. Table I is a list of important identities ; those involving half and quarter periods are concisely indicated in Tables II, III, IV.

Each chapter concludes with a set of exercises for the reader and there can be no doubt that such a presentation of elliptic functions will be very useful to those concerned with their practical application. The only possible criticism is that Mr. Bowman may have been too laconic and concise. In Table IV, for instance, the values of the argument for the middle panels must be inferred—a forward reference to this table at the end of Examples IV would be useful. Diagrams of period-parallelograms showing poles and zeros might well be added. The only misprint noted is a lacuna in equation (40) on p. 95.

C. G. P.

A Note Book in Pure Mathematics. By L. H. Clarke. Pp. 184. 8s. 6d. 1953. (Heinemann)

This book is written as a revision for students taking the General Certificate of Education at Advanced (not Scholarship) level. It contains a section on each of the subjects : Algebra, Calculus, Analytical Geometry, Pure Geometry, Trigonometry. It states results and proves all that the student would be expected to know, the proofs being well chosen, and concisely set out. For each topic there are worked examples, and short sets, most of them easy, for the student to work. At the end of each section there are revision papers,

the questions being taken from examination papers. The work is set out so clearly that the book should be most suitable for a student to use by himself for his revision.

There are few omissions. The calculus section might have included the relationship between d^2y/dx^2 and d^2x/dy^2, and corresponding results with parameters. The centre of gravity of solid figures is not considered. But the book is really very inclusive, and the trigonometry section starts with 58 formulae to be learnt, some of which most students would be content with being able to obtain rather than learn by heart.

For the immature student who is going further there should be warnings. Assumptions are not always made clear, as for instance in the differentiation of x^n, where the proof given applies for n an integer. Methods are not always those that stimulate further progress, as in the development of the exponential and logarithmic series from the expansion of $(1+x/n)^n$, with no suggestion of a calculus approach. In calculus a reference to integration as the limit of a sum would have helped the finding of centres of gravity and moments of inertia. Analytical geometry is concerned with the straight line and the individual conics in order, and there is no sign of the more general approach now coming into practice.

But for the mathematician who is not going appreciably beyond this stage, including many scientists and engineers, this book is excellent, and should provide a means of revision for which many are looking, since they can work through the book by themselves and they will appreciate the very clear exposition given by the author.

K. S. S.

Children Discover Arithmetic. An Introduction to Structural Arithmetic. By CATHERINE STERN, with a foreword by Marguerite Lehr. Pp. xxiv with frontispiece ; 295 and 12 photographs. 25s. 1953. (George Harrap & Co. Ltd.)

Dr. Stern has invented, and patented in the United States, apparatus for demonstrating, and experimenting with, the structure of our number system, with its basis of ten. She maintains that this basis can best be emphasised by using ten distinct units, corresponding to the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10. Her " one " is a cube, and the other units are blocks which occupy as much space as two, three, . . . etc. cubes, but are moved as wholes, so that the child does not count in ones, and, indeed, does not learn to count till he has handled all these units, fitting them into specially designed boxes, and measuring them against one another. These games are introduced to children at two or three years of age, and progressively other such games of fitting blocks into spaces lead them to discover more and more : the only things which must be directly taught are the number names and the figures and symbols. Because she thinks many teachers have a very superficial knowledge of arithmetic the author explains in great detail how the apparatus should be used, and, in the earlier chapters, illustrates by quoting remarks actually made by children, and describing their handling of the material : she devotes 85 pages to the learning by these means of facts about adding and subtracting numbers under ten. The three subsequent parts of the book are " Structural Techniques to Master Computation in the Range from 1 to 100 " ; " Multiplication, Division, Denominate Numbers and Fractions ", and " The Structure of our Number System ", so that the whole of what is usually the substance of primary school arithmetic, (and perhaps rather more), is to be mastered by the use of this block apparatus, and variations of it. The coins referred to are cents, nickels and dimes, and the denominate numbers do not

include as many measures as English children usually study, but, on the other hand, the treatment of ratio, percentage, and decimal fractions is made much more concrete than is our custom. Yet Dr. Stern uses her apparatus only so long as the children require it: she presents a new approach to the mathematical foundation of arithmetic instruction, but she expects a child to lay aside the material, and do his figuring in his head, as soon as he has understood each new principle. The approach is new in that it discards both rote learning without understanding, and abstracting arithmetic from life situations: its inventor claims that the learning of "pure" arithmetic by this method becomes itself a drama. In the introductory chapters the author explains the psychological basis of her wish to find a new method. and this is cleverly done, and is interesting; but, when at each step of her description of her own way of teaching she pours scorn on traditional ways, the re-iteration becomes irritating, and one feels it would be better if she let her positive exposition speak for itself without interruption. Her devices have been tried out in her own school, and in three or four others in the U.S.A. for direct learning, and with some pupils who have required remedial work in arithmetic, after beginning to learn it by other methods, but, as far as one can judge, no pupils had, up to the time of writing, completed the whole course—there are no references to individual learners in the latter part of the book, and in places it is rather laboured. Teachers will however find much to think about in the idea of structural arithmetic.

H. M. C.

Mathematics for Living. By E. R. Hamilton and C. H. J. Smith.
Book 1. Running a Home. Pp. 160. 4s. 6d. Limp; 5s. 6d. Boards. Teacher's Edition, with Introduction, Pp. vii, and Answers, Pp. 161–183.
Book 2. Earning a Living. Pp. 142. 4s. 6d. Limp; 5s. 6d. Boards. Teacher's Edition, Introduction, as above, Pp. vii; Answers, Pp. 145–161. (University of London Press Ltd.)

These are the first two books of a series of four, of which the other titles are to be "Spending a Holiday" and "Living in a Community".

The books are intended to be used under the direction of a teacher, and not usually to be worked straight through: at almost every stage there are three exercises, marked (E) (easiest), (A), and (H) (hardest). The teacher is told that interest may derive from the relation of the subject to everyday experience, or from the realisation that the subject may help the pupil to earn a living and to live his life as a citizen, or from the intrinsic interest of number and shape. The authors consider that a prime aim of mathematical teaching is to enable pupils to solve problems, and they have included many novel, yet realistic, problems as incentives to learn new mathematical skills, and to practise those already partially acquired. Rather less than half the first book is devoted to mathematics as such, including "Fun with Numbers" and "Fun with Shapes" and "Revision": interspersed among this are the chapters on buying goods for the home—food, clothes, coal, materials for knitting, carpentry, and rug-making, curtains, lino, furniture, and the house itself. It is assumed that the mathematical rules for dealing with these problems are already known, but require application and practice: the introductions to the chapters give information about tradesmen's customs, quantities, and price lists, and it is a characteristic of many of the examples that the data for the calculations must be found by referring back to these price lists. (There is a warning at the beginning of the book that prices change rapidly, but those given appear reasonably up-to-date.) At the end is a chapter called "What can we afford?" and earlier there is one on "Hunting for bargains", where the phrase "2/- in the £" is explained; there are also a few instances of money

saved by making articles at home instead of buying them ready-made, but for the most part problems of making ends meet are not included. The second book begins with the cost of travelling to work ; in a chapter on salesmanship the custom of paying a fixed wage, plus a commission on sales, is explained ; the rates of pay for miners are given ; but most of the mathematical problems considered are such as might occur in the course of doing a job, not those which concern earnings. Those on " Work in a garden ", " Decorating ", " Cookery ", and " Hairdressing " deal with costs and price lists, and are thus similar to those of Book 1 : in later chapters, however, there are suggestions for learning new mathematical methods—block graphs in " Working in a warehouse ", formulae in " Electrical work ", the decimal point in reading meters, further decimals and further formulae in " Precision engineering ", the use of π in " Garage work ", the metric system in " Laboratory work ", balance sheets in " Office work ", interest tables in " Banking' ',—and this is not an exhaustive list. As in the first book there are some chapters on pure mathematics, and plenty of revision exercises.

A few inconsistencies have been noted : it is necessary to find an area in several examples in Book 1, e.g. in Ex. 70 and in Ex. 82, but the rule for the area of a rectangle is given in Book 2, page 16 ; in Book 1, page 9, the word " addition " should be placed before " exercise 4 ". and on page 13, exercise 12 would have been better without the double lines under the sums ; on page 32 it should be stated that the angle B is a right angle. There are also some examples where it is implied, or even stated, that answers, or further drawing, should be filled in on the printed page, which is scarcely wise where books have to serve again in successive years.

But, in spite of blemishes, the books deserve to succeed in stimulating teachers in secondary schools to show initiative and freshness, as well as thoroughness, in their teaching, and to avoid the disease of mathophobia (authors' word) in their pupils. H. M. C.

Daily Life Mathematics. III. By P. F. Burns. Pp. 212. 7s. 6d. Answers, 1s. 6d. 1953. (Ginn)

Book 3 commences with a chapter on logarithms. From $2^{10} = 1024 \simeq 1000$, the author obtains log $2 \simeq 0{\cdot}3$ and he deduces the logarithms of 4, 8 and 5 from the facts that $4 = 2^2$, $8 = 2^3$ and $5 = 10/2$. A graph drawn from these values gives the logarithms of numbers between 1 and 10. The text contains only an extract from the tables of logarithms and antilogarithms so that it is necessary to supplement the text book by a set of tables. Worked examples are given which illustrate the usual processes of computation, and, later in the chapter, the construction of a slide rule is discussed. Negative characteristics are not mentioned and are avoided in the practice examples. Nevertheless, from the point of view of practical calculations, inability to cope with negative characteristics makes logarithms useless as a tool. For consider merely a simple case of division, $A \div B$. Even if A and B are both greater than 1, there is a 50% chance that $A < B$. In other words, it is quite exceptional for negative characteristics not to appear in an example involving logarithms. Near the end of the book the author gives a few worked examples involving the use of negative characteristics but there are no practice examples.

Chapter 2 deals with simple and compound interest and practical details are given of the Post Office Savings Bank and Building Societies. The next two chapters deal with the finance of Town Councils and domestic expenditure.

A chapter is devoted to the construction and evaluation of formulae. It is handled attractively and is related to concrete and interesting illustrations. Nevertheless, the author has landed himself into a difficult position. Up till

now he has avoided anything of an algebraical character. Pupils are now suddenly confronted with expressions and arguments such as

$$\tfrac{1}{2}(x+y)h=\tfrac{1}{2}xh+\tfrac{1}{2}yh, \quad D=\sqrt{(C^2+B^2)},$$
$$P^2+B^2=H^2 \quad \therefore\ P^2=H^2-B^2,$$
$$(a+b)^2=a^2+2ab+b^2, \quad a^2-b^2=(a+b)(a-b), \quad A=P\left(1+\frac{r}{100}\right)^n.$$

The reviewer would need to have some positive evidence before he could believe that the average pupil will feel at home in this chapter.

A couple of chapters are devoted to pulleys and other machines and to workshop drawings including oblique projections, plans and elevations and developments. The treatment here is good. Several chapters deal with various aspects of surveying including Earth measurements, and accounts are given of Eratosthenes' experiment and the Bedford Level experiment. The construction of a clinometer is described and examples on the use of the tangent and cosine are given. The high standard of Books 1 and 2 is well maintained. S. I.

Daily Life Mathematics. IV. By P. F. BURNS. Pp. 244. 8s. 6d. Answers. 1s. 6d. 1953. (Ginn)

This book is the last of the complete course. The first four chapters deal with ideas connected with finance and trade. The topics and their treatment give a good justification for the title of the book. The chapter on municipal and county finance deals with local government rates and loans and corresponding topics at the national level are discussed in the next chapter. The following chapter deals with national and personal insurances. The author adheres to his principle of dealing with topics and projects rather than with subjects. Thus, there is not a chapter on graphs but there are graphical illustrations depicting production and imports. Workshop drawings and plan and elevation work are extended in this Book to cover more difficult cases. The examples given are not of the artificial kind but deal with figures and structures of everyday life. Examples are given of rebattements and sections. Consideration of the sections of a cone lead to the drawing of ellipses and parabolae ; there is plenty here to occupy and interest pupils. This thoroughness is a feature of the whole of this Book and there is enough to interest and occupy the most capable of pupils. A full section is devoted to the plane table, the clinometer and the theodolite. The chapter on Earth measurements explains how to calculate the great circle distance between two points including the case when these two points have different latitudes and longitudes. In the chapter on the triangle of forces and velocities there is an explanation of how an aircraft flies and a detailed treatment of air navigation. The final chapter deals fully with sundials and nocturnals. Pupils who work through this book will have a feeling of satisfaction and achievement. Teachers in Modern Schools who have not shackled themselves to the G.C.E. examination are strongly recommended to try this set of books. S. I.

Commercial Mathematics. Vols. I and II. By J. A. TAMPKINS. Pp. 123, 122. Limp, 5s. each ; boards 6s. each. 1953. (English Universities Press)

These books are additions to the well-known Technical College Series and are the first two of a series of three designed to cover the commercial mathematics syllabuses of the U.L.C.I., the U.E.I. and the Northern Counties Examination Council.

The author, in his Preface, says " Many examining bodies now recognise the importance of algebraic and graphical methods . . . in the solution of commercial problems ". This statement forms the keynote to the work of the two books for one finds algebraic notation introduced as early as Chapter 5, Vol. I and followed closely by graphs of statistics in Chapter 8. From this point onwards the more elegant methods of algebra are used whenever possible, and, at least so far as the reviewer's experience goes, to a much greater extent and with a more refreshing approach than heretofore.

Such criticisms as can be made are of a minor nature. Where does one find the first year student who can reduce to its lowest terms the fraction $\frac{399}{1045}$ and what useful purpose is served by this drudgery? For the decimalisation of money the student is given a list of equivalents and later is told that for some purposes it is not sufficiently accurate to work to three places of decimals. It is the reviewer's experience that there is less confusion in the student mind if he is taught from the beginning to decimalise money by the successive reduction of farthings to pence, pence to shillings, etc.

Chapter 9 of Vol. I deals with ratio and proportion and would seem to be a good point at which to introduce the graph of $y=mx+c$. However, the straight line graph is reserved for Vol. II, with the consequence that the student is asked to solve simultaneous equations graphically in Vol. II having learnt to solve them much more easily and more accurately in Vol. I.

While one has to agree with the author that the use of logarithmic graph paper would considerably reduce the complexity of the work on the graphs of the type $y=ax^n$, one finds oneself asking the question " What about the availability of such paper? "

On the credit side are the author's approach to directed numbers, his many commercial applications for simple equations, for formula manipulation, for simultaneous equations, and, also, the number and variety of the worked examples in the text. If for these characteristics alone, the books should prove assets to both teachers and students of the subject. G. W. H.

Arithmetic Made Easy. By W. Haydn Richards. Book I. Part I. Pp. 88. Book I. Part II. Pp. 78 and two charts. 3s. each. 1953. (Harrap)

This is stated to be a four years' course for juniors. It reduces calculation to a series of rules, stated precisely, and each followed by practice exercises on it ; but there is no attempt to appeal to the interest of beginners, or to give any explanation of the rules. In the hands of the inventor of the methods, teaching on these lines may be very effective, but in their printed form the early stages seem very dry bones. Moreover, it is doubtful whether a seven-year-old would have the reading ability to follow the directions of, say, page 9, which are in rather small type for this age. Some of the " sums with words " are too much classified, though the words are well varied : e.g. a child might work exercise 37 correctly, by picking out the numbers, and dividing each by 2, without reading any of the questions, or thinking what he was doing. Exercise 130 seems intended to deal with this point, though surely somewhat belatedly, and some of the other exercises in Part II are more realistic. The Self-help Charts are good and clear, and the miscellaneous drill practices are ingeniously arranged to stimulate effort. In spite of the very careful analysis of the elementary steps, there are some oversights which may cause trouble : e.g., on page 10, $3+1=4$ becomes

$$\begin{array}{r} 1 \\ +3 \\ \hline \underline{4} \end{array} \quad \text{(not} \quad \begin{array}{r} 3 \\ +1 \\ \hline \underline{4} \end{array}\text{)}$$

but on page 15

$3-1=2$ becomes $\begin{array}{r} 3 \\ -1 \\ \hline 2 \\ \hline \end{array}$ and there is no comment on the change of order.

Again is it correct to say " The sign $(-)$ means *from*? " If the word is used at all surely we should say " From 3 take 1 ". H. M. C.

Contributions to the Theory of Games. Volume II. Edited by H. W. Kuhn and A. W. Tucker. Pp. viii, 395. Annals of Mathematics Studies No. 28. 25s. 1953. (Princeton University Press: London, Geoffrey Cumberlege)

Annals of Mathematics Studies No. 24, which had the same title as the book now to be considered, was reviewed in the *Math. Gaz.* (vol. 37, 134, 1953) and the publication of a second volume with 21 papers, so shortly afterwards, suggests that it might be appropriate to survey here, as briefly as possible, the present state of the subject. The first volume contained in its Preface a list of 14 unsolved problems, which provided some sort of briefing for workers in this field. Not all these problems are tackled in the second book, but some have been investigated, not without success.

While the earlier book was divided into two Parts: Finite Games, and Infinite Games, the present book contains four: Parts I and II deal with finite and infinite zero-sum two-person games respectively, and Part IV with general n-person games, which were hardly mentioned on the previous occasion. Part III treats games in extensive form. This classification provides a guide to recent investigations in game theory.

One might start conveniently from the theory of the zero-sum two-person game with a finite number of strategies. The most important problem in this connection remains that of a satisfactory method for computing optimal strategies and the value of a game (which is, in this case, known to exist). Various approaches are possible and many have been explored: reduction of a game to another seemingly unrelated one, whose solution is equivalent to that of the first; determination of pure strategies which must form part of the mixed strategies in the solution; illustrative examples giving solutions for such well known games as Poker, though of a simplified variant. It does not appear that the possibilities of attacking the computational problem have been exhausted.

As distinct from the games just mentioned, we have infinite games, general (non-zero-sum) games, and n-person games. It has been shown, *inter alia* in the present book, that infinite games present the most curious features, compared with analogous finite games. For instance, they need not have a solution, even in the two-player case. It is thus natural to ask for classes of infinite games for which at least some results of finite games carry over. Part II of our book contains two papers in which an explicit method is given for the solution of " games of timing ". These are duels in which the opponents must decide on when to use their resources. Each player gains by using them later, provided he uses them earlier than his opponent.

An extension into another direction are the n-person games. Here the situation is rather unsatisfactory, mainly because no agreement has been reached on what should be called a " solution ". We must distinguish between cooperative games, studied by von Neumann and Morgenstern, which allow the players to form coalitions, and non-cooperative games, introduced by Nash; attempts at reducing the former to the latter are not entirely successful. The following statements summarise our present knowledge.

Von Neumann and Morgenstern define a solution as a set of repartitions of payments with certain properties, which space does not permit to enumerate here. It is not known whether all cooperative games have, in fact, a solution in this sense. On the other hand, Nash defines " equilibrium points " as such collections of (mixed) strategies of the players, that each player's strategy maximises his pay-off, if the strategies of the others remain fixed. A solution is, then, a set of equilibrium points satisfying the further condition that, if we select for each player one of his strategies which belong to some equilibrium point, the resulting set is also an equilibrium point of the set. All finite games have at least one equilibrium point, but not all have a solution.

Studies with extensive games are concerned with a different aspect of the theory. It is known that the original results of von Neumann related mainly to the normalised game, i.e. that described by the pay-off matrix, which tabulates the payments as dependent on the strategies chosen by the players. There exists one result, though, which takes explicit account of the succession of moves in a play. This is the theorem that a two-person game with perfect information (e.g. chess, but not card games), has a saddle point in pure strategies. The central concept here is that of information patterns, i.e. rules about what each player knows about previous moves. The theorem just mentioned remains valid for Nash's equilibrium points in (finite) n-person games. The present volume includes more investigations in this field.

It is easily seen that the subject is by no means dying of lack of interest, and we can look forward to more collections of this type. The bibliography, supplementing that of the first volume, is already out of date. S. VAJDA.

The Fourth Mental Measurements Year-Book. Edited by O. K. BUROS. Pp. xxiv, 1163. 1953. (Gryphon Press, New Jersey)

The *Third Mental Measurements Year-book* was noticed in the May number of the *Mathematical Gazette* 1949, and previously publications of this series were referred to.

The new volume follows the lines of the 1949 publication, being divided into two main parts dealing with (*a*) Tests and Reviews, (*b*) Books and Reviews.

It covers the period 1948 through 1951 and lists 793 tests, 596 original test reviews by 308 reviewers, 53 excerpts from test reviews in 15 journals and 4,417 references on the " construction, validity, use and limitations, of specific tests ". The Section " Book and Reviews " lists 429 books on measurements, and 758 excerpts from book reviews in 121 journals.

The aim, as previously, is to provide an up to date review of all recent work on mental testing, so that a more critical attitude towards testing techniques may be adopted, and more suitable tests be selected for the needs they are required to satisfy.

The book is beautifully and strongly produced, and is arranged to be easy to use. Anyone working in this field should find it an invaluable aid.

F. W. W.

Communication Theory. Papers read at a symposium, 1952. Edited by WILLIS JACKSON. Pp. xii, 532. 65. 1953. (Butterworth)

The new subject which Professor Wiener has christened " Cybernetics " can be described as the unification of three recently developed branches of engineering, with particular reference to their application to problems in neurology and sociology. These branches are servomechanisms, automatic computation and communication theory. The first two received considerable stimulation during the late war, but the third appears to have flowered more recently. The present book shows however that it is bearing abundant fruit.

Perhaps the most significant contribution to communication theory is contained in a paper by Shannon (" The Mathematical Theory of Communication ", *Bell Syst. Tech. J.* (1948)), which has been reprinted in book form. In a review of this book in the *Gazette* (Vol. XXXIV, No. 310, p. 312) the uncertainty as to where this fascinating subject leads is commented upon. It is certainly true that a reader of Shannon's paper cannot but wonder at the wide gap between the basic mathematical theory and possible useful engineering applications. The papers presented at this symposium provide a demonstration of the way in which this gap can be bridged. As with the other two subjects mentioned above, the engineers appear to have turned to good account an ingenious mathematical philosophy.

The symposium was introduced by a paper from D. Gabor entitled " A Summary of Communication Theory ". This provides a very useful commentary on Shannon's paper, which although soundly written is not, in the language of the subject, blessed with overmuch " redundancy ", and in places has to be " decoded " rather than read.

The first main section of papers is headed " Transmission Systems and Coding ", a feature of which is, in general, the application of Shannon's and other theories to alternative modulation systems in radio communication. The second section is devoted to the effect of noise. Correlation techniques play a large part here. Three shorter sections follow. These are concerned with transmission channels, television and hearing. Next comes an extensive and important section on the analysis and transmission of speech. The volume ends with four papers of a miscellaneous character and an interesting concluding discussion. This provides a summary of what has been achieved so far, and perhaps more important, an indication of some of the problems awaiting solution.

Each paper is followed by an account of the discussion that followed its presentation at the symposium. These discussions were obviously an integral part of the proceedings and they enhance considerably the value of the present volume.

A word of praise is due to the editors and publishers of this work. There is no obvious evidence of the difficulties that must have been met in producing, in a comparatively short time, a work to which there are nearly fifty contributors from both sides of the Atlantic. The only criticism that can be offered concerns the absence of an index of any kind. B. M. Brown.

Leitfaden der Nomographie. By W. Meyer zur Capellen. Pp. 178 with 203 figures. D.M. 17.40. 1953. (Berlin; Springer-Verlag)

This book gives a very full account of the subject in a comparatively small volume. After a brief introduction there are two main sections, on the theoretical bases of the subject and the construction of charts and nomograms for particular equations.

The theoretical section, whilst not exhaustive, includes, or gives pointers to, everything that is likely to be required. It opens with a discussion of the graphical representations of functions in various co-ordinate systems, the construction of double-sided scales for functions of one variable, slide-rules, the use of logarithmic, polar and triangular graph papers, intersection charts for functions of two variables and the combination of charts, including moveable grids, for relations between more than three variables. Then follow alignment charts. For relations between three variables the author discusses in order of increasing difficulty, three parallel straight lines, three concurrent straight lines, N-type nomograms, two straight lines and a curve, circular nomograms and the general case of three curves. The errors introduced by

inaccurate construction are examined. For four or more variables the author discusses multiple alignment nomograms (with straight pivot lines only), set-square index nomograms, grid nomograms, described as combinations of charts and alignment nomograms, and a particular case of a nomogram with moveable scales. Double nomograms are not mentioned but examples are given later. The section concludes with interesting discussions on the transformation of nomograms by projection and on the duality between intersection charts and alignment nomograms. Although it would follow naturally from this, no mention is made of the occasional necessity to transfer from one scale to another along a tangent to a curve.

The next section, which describes the construction of charts and nomograms for a great variety of equations, follows the same sequence. It is well illustrated by a large number of small but clear figures.

There is an adequate index, and a bibliography, mainly of German and French publications. C. V. GREGG.

An Introduction to Statistics. By CHARLES E. CLARK. Pp. x, 266. 34s. 1953. (Chapman and Hall)

The author of this book is Associate Professor of Mathematics at Emory University, and the author of *College Mathematics*. The present book is " a first book for beginners in statistics " and is intended to develop an appreciation of the nature and significance of statistical inference.

Let us consider it first from the point of view of the mathematician. He will be struck by its lack of rigour. Phrases like " nearly equal to ", " roughly normal ", " not sharply defined ", " differ so little ", " in a vague way ", constantly recur. The writer frankly says at certain stages that there are " difficulties that we shall not face ", " proofs that we do not care to undertake ", whilst some of the statements and " proofs " are little more than suggestions. We must look for the value of the book in its serviceability for the job the writer undertakes.

On the whole, it is probable that the novice will get a good idea of some of the points that a statistician has in mind when he applies his various tests. The distinction between " empirical ", " statistical ", and " a priori " probabilities may make good teaching points, and the idea of " roughly normal " distributions avoids much more advanced work. The first five and a half chapters are devoted to first ideas of combinations, etc., and of the various probabilities, with examples of means and of standard deviations. About half way through the book the idea of confidence levels is introduced, and much of the work after this is built around this, with first ideas of analysis of variance and of chi-squared. The last chapter, Ch. 9, turns from statistical inference back to the description and analysis of empirical data by a simple consideration of correlation : the level of algebra needed is perhaps indicated by pointing out that on p. 216 it is remarked that " the graph of $y = Ax + B$ will be a straight line. This result ... we shall assume without proof ".

On p. 219 the author begins to set out answers and solutions for odd-numbered exercises. The exercises have occurred after nearly every section of each chapter, and are often based on a rich variety of material culled from various sources. The reviewer was interested to see that some figures about attendances at his school that were published in 1938 in England were used as an example for a chi-squared test. The solutions are usually very full, and we have not found any errors in them, apart from some vague and non-rigorous turns of phrase and argument. Occasionally the answers incorporate further questions. I note that the method of using tallies is first given in a solution of Section 4.2, although this method is not explained in the text until the last chapter.

At the end of the book are six tables, of squares, of logs, of normal distribution areas, for t, F, and chi-squared. Some of these are, I think, from plates that I have seen before : I have not checked any of them. It is, however, I think, unfortunate that the abacs of p. 124 (for t), of pp. 165 and 166 (for F), and p. 185 (for χ^2) are placed in the text and not put with the tables at the end of the book. But the book as a whole is well printed and the misprints are very scarce and trivial. It is one of the *Wiley Publications in Statistics* and, like the others that we have seen—*Sequential Analysis* (Wald, M. G., 33 (303) 66–68), *Experimental Designs* (Cochran and Cox, M. G., 36 (315) 78–79), and their *Introduction to the Theory of Probability and Statistics* (Arley and Buch, M. G., 35 (314) 288–289) in their *Applied Mathematics Series*, is attractively produced. Although it does not attempt to be more than a very simple introduction it will probably prove to fill a gap and be of value to a certain class of reader in this country. FRANK SANDON.

Complex Variable Theory and Transform Calculus. By N. W. McLACHLAN. Second edition. Pp. xi, 388. 55s. 1953. (Cambridge University Press)

Though the title has been changed slightly, this is the second edition of a book originally published in 1939, and comprehensively reviewed in the *Mathematical Gazette*, Vol. XXIII, pp. 427–429. The change of title is, perhaps, judicious, for the book has been largely re-written.

Part I consists of Complex Variable Theory. Though none could accuse Dr. McLachlan of wishing to take the mathematics out of Engineering Mathematics, this part of the first edition was criticised for lack of rigour. Now that this criticism has been largly met, one wonders whether the book could have been made cheaper by leaving out Part I altogether, and referring the intending reader to standard mathematical works on the Complex Variable. However, it must be said that here there is a wealth of illustrative example not usually found in purely mathematical books, and this feature should prove very helpful to the non-mathematician.

Parts II and III are on Operational Theory and Technical Applications respectively. Part III will prove invaluable to the technologist and applied mathematician. A criticism of Part III in the Gazette review of the first edition was that Dr. McLachlan used the Inversion Integral to the almost complete exclusion of operational forms. This new edition contains a list of fifty five transforms, and, in addition to the complex integral, reference is made wherever possible to the appropriate transform for the purpose of inversion.

The book contains a hundred and two examples, and anyone able to do all these would be a very useful practitioner in the field of Operational Calculus.
M. HUTTON.

Faster than Thought, ed. by B. V. BOWDEN. Pp. 416. 35s. 1953. (Pitman)

This book has set a high standard as the first comprehensive, readable, informative account of modern developments in automatic digital computing machines. It is primarily a collection of articles by a large number of authors, each contributing an account of his own work or field. The accounts are for the most part clear and well written, and together they provide a remarkably balanced and complete picture of the present position in this country.

There are three parts to the book ; the first is an introduction to the subject, written mostly by the editor himself. His writing is witty, entertaining and liberally sprinkled with delicious anecdotes, yet always well aimed and penetrating. The historical chapter which starts the book is particularly delightful and will be read with interest by everyone connected with automatic computers

as well as by others not yet acquainted with the subject. The value of the book has been greatly increased by including, in an appendix, a reprint of the classic translation with notes by Lady Lovelace of Menebrea's paper on the Analytical Engine of Babbage.

The second chapter is one of the less successful in the book. It attempts to lead a novice, in a few pages, from the notion of a valve to the intricacies of the actual circuits used in an electronic digital computer ; this is unfortunately impossible. However, most of the chapter will be of interest to those with some knowledge of electronics, and those without will be glad to know that circuit details are quite irrelevant to an understanding of the principles of computers. In the rest of the book, circuits are described only in general terms. The next two chapters are in fact excellent non-technical commentaries on the arts of designing and building computers.

Chapter 5 is also rather too condensed, which is a pity since it deals with the more essential question of how one makes effective use of these machines. The facts are all there, but the ideas are so novel that some readers may not be able to see the whole picture.

The second part of the book is a valuable collection of short descriptions of existing British machines, with one chapter summarising progress in the United States. The third part will be of most interest to potential users of electronic computers ; it discusses applications of these machines to a variety of different types of problems. Particularly interesting are the chapters on meteorology, engineering, business, and games. The final chapter on " Thought and Machine Processes " is no more conclusive, but much more interesting, than most of the articles which have appeared recently on the subject.

Altogether, this book is thoroughly recommended for browsing to anyone with any kind of interest in the subject, and many will find it a useful source of general information. S. Gill.

Höhere Mathematik für Mathematiker, Physiker und Ingenieure. By R. Rothe. Vol. II. 9th edition. Pp. 210. D.M. 6.50. Vol. III. Pp. 242. D.M. 8.20. Vol. VI. Prepared by I. Szabo. Pp. 251. D.M. 17.60. Teubners Mathematische Leitfäden, 22, 23, 45. 1953. (Teubner, Stuttgart)

With Vol. II of Rothe's book in a ninth edition, and Vol. III in a sixth edition, repetition of earlier recommendation in the *Gazette* would seem superfluous. The lucid and informative account of integration, infinite series, contour integrals, determinants, vectors, curvilinear and multiple integrals, and differential equations has clearly proved its worth. In the revision of volume III, the section on partial differential equations has gone to Vol. VI, and in its place there is a short account of numerical methods, including those of Runge-Kutta and Adams.

Vol. VI is an extension of the original design, due to the rapid developments in the applications of mathematics during the last thirty years. The author, Dr. Szabo, remarks that if an applied mathematician, physicist or engineer wishes to read and understand papers in the *Zeitschrift für angewandte Mathematik*, *Ingenieur-Archiv*, *Zeitschrift für Physik*, *Annalen der Physik*, he must now know something about the theory of differential and integral equations and be able to handle special functions (Bessel, Legendre, Mathieu, Whittaker). Thus he gives more function theory, including asymptotic series, something about the theory of differential equations, a good deal about those linear equations which determine the special functions of most importance, and a very little about partial differential equations. All this is heavy going for the average engineer, and he will complain that much of it seems far removed

from his own domain ; even the impressive examples of the uses of Mathieu functions may not reconcile him to a study of that untidy and sprawling region. Dr. Szabo is not to blame ; his account is as clear and straightforward as it could well be. The plain fact is that physicists and engineers resent the mathematics that physics and engineering now demand, and tend to blame the mathematician for a situation which is none of his making. Those who determine to make the best of what may seem to them to be a bad job will find Szabo's account stiff but helpful. T. A. A. B.

Recherches Arithmétiques. By C. F. Gauss. Reprint of the translation by A. C. M. Poullet-Delisle, 1807. Pp. xxii, 502. 1953. (Librairie scientifique et technique, 7 Rue Racine, Paris)

The re-issue of a classic, even in translation, cannot be reviewed. This is the *Principia* of number-theory, and the prince of mathematicians needs neither herald nor public relations officer.

The reproduction is *fac simile* ; have Messrs. Blanchard exhumed, from some forgotten nook in the Quai des Augustins, a long-lost stock of sheets of this translation? The whole production brings back the air of the First Empire, and one rubs one's eyes at the sight of the 1953 date on the title-page. Even the dedication, to Gauss' patron and benefactor, Charles William Ferdinand, Duke of Brunswick, conjures thoughts of the guns of Valmy and the debacle of Jena and Auerstadt. Or again, one may wonder why a French translation appeared within six years of the original publication ; the story of a bookseller's failure causing a scarcity which prevented Eisenstein from obtaining a copy and induced the cautious Dirichlet to sleep with his copy under his pillow, may provide the explanation.

Number-theorists will not fail to recognise Messrs. Blanchard's homage to the master. T. A. A. B.

Einführung in die Analytische Geometrie. By G. Kowalewski. 4th edition. Pp. 364. 1953. (W. de Gruyter, Berlin)

This is a very nicely produced fourth edition of a classical text first published in 1910. The steady sale of the book since then is as good a recommendation as any reviewer could add. As the book is intended to be an introduction its treatment of some topics (for example, the angle between two lines in space, imaginary points) leaves something to be desired, but, starting as it does from the beginning it still goes further into parts of the subject than many English text-books. D. B. S.

Praktische Mathematik. By H. von Sanden. 3rd edition. Pp. 128. D.M. 3.20. 1953. (Teubner, Leipzig)

This is an enlarged edition of a textbook of numerical and graphical methods written for the Engineering Department of the Technische Hochschule at Hanover. The first of the six sections deals with graphical and mechanical methods ; the second with approximations based on Taylor's theorem, including Newton's method ; the third devotes 20 pages to finite difference methods of integration, differentiation and interpolation. The fourth and fifth together comprise forty per cent. of the whole and deal with Statistics and Least Squares methods, including the use of probability paper. The last section is devoted to a short account of practical harmonic analysis based on a stencil method rather different from that of Milne-Thomson (unfortunately

the full set of stencils is not shown and so it is not possible to assess the relative merits of the two systems).

A noteworthy feature of the whole book is the emphasis on the accuracy to be expected when data are subject to errors.

J. C. W. D.

Elementary Differential Equations. By L. M. KELLS. 4th edition. Pp. x, 266. $4. 1954. (McGraw-Hill)

That this is the fourth edition since 1932 suggests that the practical flavour imparted to his book by the Professor of Mathematics at the U.S. Naval Academy is very welcome to the young technician. Previous reviews have dealt adequately with the main aim and content; thus it is enough to say that in the new edition there has been a revision of the text, a more careful grading of and some additions to the stocks of examples, and a change of emphasis in treating partial differential equations. My main complaint is that the numerical solution of differential equations should have, in a book of this size, either a full chapter or no mention at all; four pages is merely derisory.

T. A. A. B.

Cardano: The Gambling Scholar. By O. ORE. Pp. xiv, 249. 25s. 1953. (Princeton University Press; London, Geoffrey Cumberlege)

Cardan, typical versatile Renaissance rascal, owing his mathematical fame to what he stole from Tartaglia: to this widely-accepted characterisation, Professor Ore applies a searching corrective, emphasising the versatility but denying the rascality, presenting the Tartaglia episode in detail so that we may ourselves judge how far Cardan was at fault, and finally claiming Cardan as a pioneer in probability theory, preceding by fully a century the accepted founders, Pascal, Fermat and de Méré.

After a brief account of Cardan's eventful life and an analysis of his inquisitive, appetitive, versatile, turbulent spirit, there is a long chapter on the academic controversies in which he was so frequently embroiled, particularly the long drawn out campaign waged by Cardan and Ferrari against Tartaglia. This almost of necessity is tedious reading, but Ore argues that though Cardan broke faith with Tartaglia, thus giving genuine grounds for serious complaint, he was contending for the modern view that a scholar must publish his findings against the medieval view that a discovery was private property. The case is well argued, and rendered highly plausible if not entirely convincing. In another idiom, we might say that there was a good deal of dirty in-fighting.

But perhaps the most interesting chapters are the last two, together with the very valuable appendix in which Professor Gould of Purdue University gives a full translation of Cardan's *Book on games of chance*. It is well known that Cardan was a great gambler but it has often been said that his book contributes nothing to the establishment of a theory of probability. This is clearly an under-estimate; the steps are fumbling, the direction of advance not always clear, but steps are made and new ground is won. Even with Ore's careful analysis, some sections remain to me unintelligible, but more than enough can be found to establish Cardan's right to rank as a pioneer. The foundations of the theory are matters for vigorous and bitter controversy even today, so we need not be surprised to find that Cardan made false starts, tried methods not consistent with one another, failed to establish a clear, precise and unchallengeable axiomatic system. On Ore's showing, what he did is sufficient to place him firmly among the founding fathers. T. A. A. B.

Introduction to Symbolic Logic. By A. H. Basson and D. J. O'Connor. Pp. viii, 169. 7s. 6d. 1953. (University Tutorial Press)

Introductions to a subject are of many kinds. Some books under the title of an introduction seek to cover the whole range of human knowledge in their chosen fields, and what they are introductions to are presumably the frontiers of discovery. Others survey a broad landscape and chart a few of the main roads. This little book however does neither of these things but gives a detailed account of a small fragment of symbolic logic, an account which combines lucidity, style and fidelity to truth to a high degree.

The ground covered is the propositional calculus, with detailed proofs of the independence, consistency and completeness of the axioms, and the concepts of the predicate calculus (with special emphasis on the monadic predicate calculus) stopping short of the formulation of an axiom system for the predicate calculus. There is also an appendix on the algebra of classes and the syllogism.

In outlook and treatment the book belongs to the nineteen-twenties rather than to the fifties. There is no mention of Herbrand's deduction theorem, not even for the propositional calculus, and not a hint of Gentzen's logic of natural inference ; the reader is given the impression that logic is unique and there is no comparison with other systems such as many-valued logic or intuitionistic logic. The introduction of numerals 1, 0 to denote truth and falsehood is pointless, since there is no arithmetical analysis of propositions. A good example of such an arithmetical decision procedure for the propositional calculus which does not appear to be sufficiently widely known, is that in which propositions are represented by variables $p, q, r, \ldots$ taking only the values 0, 1, negation, disjunction, conjunction and implication being denoted by the functions $1-p$, pq, $p+q-pq$ and $(1-p)q$ respectively. A tautology is a proposition whose representing function is identically zero. For instance, since $(1-p)p=0$ whether p is 0 or 1, and

$$\{1-(1-p)q\}(1-pr)qr=0 \quad \text{for all values 0, 1 of } p, q, r$$

therefore the propositions " P or not-P " and

" (P implies Q) implies {(P or R) implies (Q or R)} "

are tautologies. To prove

{(P implies Q) implies R} implies {(R implies P) implies (S implies P)}

it suffices to observe that *if*

$$[1-\{(1-(1-p)q\}r]\{1-(1-r)p\}(1-s)p=1$$

then each factor on the left has the value unity, so that $p=1$, $s=0$, $r=1$ and $1-r=1$, which is impossible.

The procedure which the authors call derivations by substitution is also clarified by this analysis. One of the examples worked in the book is to show that S can be derived from the four propositions (i) (G or F) implies (J or S), (ii) (F or J) implies P, (iii) not P, (iv) G ; this is equivalent to deriving the equation $s=0$ from the equations

$$(1-fg)js=0, \quad (1-fj)p=0, \quad 1-p=0, \quad g=0.$$

From the second and third equations we derive $1-fj=0$, whence $f=j=1$, and so from the first and fourth equations, $s=0$. By classical methods this derivation takes several pages of work.

A number of comparatively minor misconceptions in the book perhaps merit comment. We are told (on p. 2) that if $x^2=4$ then x is a variable which can take either of the two values $+2$ and -2, and if $x+y=7$, x and y are

variables which can range over the integers 0 to 7. Why is x allowed to be negative in one case and not in the other? And cannot x take *all* values, some of which make $x^2=4$ true and some make the equation false? In fact to say that if $x^2=4$ then x takes either the value $+2$ or the value -2 is only to say that the implication

$$(x^2=4) \text{ implies } (x=2) \text{ or } (x=-2)$$

is true, and here x is a free variable for which any number may be substituted. We do not need to introduce a special category of variable to handle equations. On p. 9 we are asked to consider what mathematical calculation would be like if we lacked special signs such as multiplication and addition signs, but in fact the lack of these special signs would cause no difficulty in calculation. It was the invention of decimal representation which made calculation easy, not the introduction of special signs for addition and multiplication.

To say (p. 19) that if we have the expression $y=3x+2$ then y is a function of x, is rather misleading. In the relation $y=3x+2$, x and y are both variables which may take any values, some of which will satisfy the relation and others will not; it is true that if we substitute the function $3x+2$, for y then the resulting predicate holds for all values of x. but this does not entitle us to confuse the *relation* with the *function*; and in fact the distinction between the two is a very important one.

It does not follow (p. 22) that, because not-not-p and p have the same truth values, not-not-p and p are the same proposition, but only that they are equivalent, nor is it true to say (p. 28) that " p and q " and " not(not-p or not-q) " are equivalent by definition, since in fact conjunction, disjunction and negation have all been defined by truth tables. And the remark (on p. 29) that not all of the sixteen possible truth functions (of not more than two arguments) are logically interesting, does not seem to make sense; the point is surely that the sixteen are not all independent and may all be expressed in terms of conjunction, disjunction and negation (or in terms of Sheffer's stroke).

In spite of its limitations the book may be safely recommended to beginners, for it is attractively written, very easy to read, informative, and, on most topics, accurate and reliable. R. L. GOODSTEIN.

An Introduction to Symbolic Logic. By S. K. LANGER. 2nd edition. Pp. 367. Paper $1.60; cloth $3.50. 1953. (Dover, New York)

The first edition of Mrs. Langer's book was published in 1937, when the number of texts on symbolic logic was small indeed; the situation has changed considerably in sixteen years, with Rosenbloom, Tarski, Quine, Rosser, Kleene, to mention only a few names, all available and covering all levels, from the novice to the expert. This new edition differs from the first only in the correction of errors, the addition of a short appendix on the truth-table method, and the inclusion of a few recent titles in the helpfully-annotated list of books for further reading. Thus for those who wish to know what mathematical logic is doing today, this book will not suffice; but it remains one of the clearest and simplest introductions to a subject which is very much alive. The style is easy, symbolism is introduced gradually, and the intelligent non-mathematician should have no difficulty in following the arguments; altogether a very useful primer. T. A. A. B.

FOR SALE

1. *Mathematical Gazette*, Vol. XXVIII, Feb. 1944, No. 278—Vol. XXXVII, Dec. 1953, No. 322. With index parts.

Offers to L. A. W. Jones, Berkhamsted School, Herts.

2. *Mathematical Gazette*, Vol. X, No. 144—Vol. XXXVII, No. 322. Lacking Nos. 161–5, 192, 212, 296, 308, 316.

Offers to Miss M. C. Nightingale, 16 Whitecross Street, Barton-on-Humber, Lincs.

THE LIVERPOOL MATHEMATICAL SOCIETY

Liverpool Branch of the Mathematical Association

Report for the Session 1952–1953

Officers.—*President*, Dr. C. W. Jones; *Vice-President*, Mr. J. Kershaw; *Secretary*, Dr. G. R. Baldock; *Acting Secretary*, Mr. E. J. Watson; *Treasurer*, Mr. L. Sowerby; *Auditor*, Miss W. Taylor; *Committee:* Dr. W. B. Bonnor, Mr. E. D. Camier, Mr. W. E. Egner, Miss M. Greig, Mr. A. C. Paris, Professor L. Rosenhead, Mr. A. T. F. Nice (*ex officio*).

Report of Meetings

20th October, 1952. The subject of Dr. C. W. Jones's Presidential Address was " Relaxation Oscillations ". Dr. Jones explained the use of the phase plane to describe the motion of linear oscillators and considered the effects of positive and negative damping on the same mechanical or electrical system. The sinusoidal and the " square " relaxation oscillations could be regarded as extremes of behaviour in certain systems and it was possible to change from one to the other by variation of a parameter such as an electrical resistance. Everyday examples of relaxation oscillations were mentioned and the lecture was illustrated by an experiment with an oscillating multivibrator circuit arranged by Mr. J. Kershaw.

10th November, 1952. A discussion was held on " The Teaching of Statistics in Schools ", based on a report of the Royal Statistical Society with the same title. Mr. R. L. Plackett (Liverpool University), who opened the discussion, said that the questions to be considered were whether statistics should be taught in schools, and if so, what should be taught. He showed the syllabus suggested for the General Certificate of Education in the Royal Statistical Society's report and gave a list of useful text books and reference works. Several members of the Society spoke in the discussion and gave accounts of their experience in teaching statistics. It was pointed out that statistical ideas were important in many subjects, both in and out of school.

1st December, 1952. Professor R. A. Rankin (Birmingham University) addressed the Society on " The Closest Packing of Circles and Spheres ". A packing of circles was defined as an arrangement of non-overlapping equal circles, and the packing constant as the limit of the ratio of the total area of the circles within a large square to the area of the square when the sides of the square tend to infinity. The problem considered was to find ρ_2, the greatest possible packing constant. Extensions of this problem included packings of spheres and hyperspheres, of circles of different sizes and of circles on a sphere. A sketch was given of the proof that $\rho_2=\pi/\sqrt{12}=0{\cdot}906$, where the corresponding packing has each circle in contact with six others.

2nd February, 1953. Professor A. G. Walker (Liverpool University) spoke to the Society on " Distances in the Universe ", and traced the methods used for estimating astronomical distances from the nearest stars to the furthest nebulae. The motion of the earth about the sun causes the nearer stars to have an apparent change of position, and from the parallax of such a star its

distance can be calculated. If we can estimate the absolute brightness of a more remote star, its distance can be found from its observed apparent brightness. The most useful stars for this purpose are the Cepheid variables, whose period of fluctuation is related to the absolute brightness. In this way the distances of the nearer extragalactic nebulae can be found. The light from these nebulae has a " red-shift " in its spectrum which, if interpreted as a Doppler effect, indicates that the nebulae are receding, and the velocity of recession is found to increase with the distance of the nebula observed.

16*th March*, 1953. In giving the Society " Some Thoughts on Text-books " Dr. W. L. Ferrar (Hertford College, Oxford) considered some of the difficulties experienced in writing a text-book for use in schools. The author had to decide what the subject was about, how to present it for the readers he was addressing, and what the range of the book should be. Dr. Ferrar illustrated his argument with examples from his own text-books and other well-known works.

11*th May*, 1953. The Treasurer's Interim Report was presented at the Annual General Meeting and discussed. Elections were held for the Officers and Committee members of the Society for the Session 1953–1954.

After the business meeting Dr. W. S. Owen (Liverpool University) spoke on " American Education of Technologists ". Dr. Owen gave a brief account of the types of Universities in the United States and then described the educational system of the Massachusetts Institute of Technology. Undergraduates were required to spend a fifth of their time on the humanities and might work in their spare time. The graduate school produced research workers of a high average standard. Dr. Owen's lecture was illustrated by coloured slides.

E. J. WATSON
Acting Secretary

BOOKS FOR REVIEW.

A. Basson and D. J. O'Connor. *Introduction to symbolic logic.* Pp. viii, 169. 7s. 6d. 1953. (University Tutorial Press)

W. G. Bickley. *Bessel functions and formulae.* Pp. xxx-xl. 3s. 6d. 1953. Extracted from the British Association Mathematical Tables, Vol. X, part II. (Cambridge University Press)

L. H. Clarke. *A General Certificate calculus.* Pp. viii, 222. 10s. 6d. 1953. (Heinemann)

R. G. Cooke. *Linear operators.* Pp. xii, 454. 52s. 6d. 1953. (Macmillan)

J. Delhaye. *Astronomie stellaire.* Pp. 212. 250 fr. 1953. (Colin, Paris)

C. S. Draper, W. McKay and S. Lees. *Instrument engineering. II. Methods for associating mathematical solutions with common forms.* Pp. xxviii, 827. 120s. 1953. (McGraw-Hill)

J. L. E. Dreyer. *A history of astronomy from Thales to Kepler.* (Rep.) Pp. 438. $1.95 ; cloth $3.95. 1953. (Dover Co., New York)

M. L. Dubreil-Jacotin, L. Lesieur et R. Croisot. *Leçons sur la théorie des treillis des structures algébriques ordonnées et des treillis géométriques.* Pp. viii, 385. 5500 fr. 1953. Cahiers scientifiques, 21. (Gauthier-Villars, Paris)

R. L. Goodstein and E. J. F. Primrose. *Axiomatic projective geometry.* Pp. xi, 140. 15s. 1953. (University College, Leicester)

S. L. Green. *Advanced level pure mathematics.* **I.** *Algebraic plane geometry.* Pp. 128, iv. 6s. *II. Pure geometry and trigonometry.* Pp. 129-264, iv. 6s. 1953. (University Tutorial Press)

E. R. Hamilton and C. H. J. Smith. *Mathematics for living. I. Running a house.* Pp. 160. *II. Earning a living.* Pp. 142. Limp, 4s. 6d. ; boards, 5s. 6d. each. Teacher's edition, with answers and introductory chapter, 6s. 6d. each. 1953. (University of London Press)

Sir Thomas Heath. *The works of Archimedes. The " Method " of Archimedes.* (Rep.) Pp. clxxxvi, 326, 51. $1.95 ; cloth $4.95. 1953. (Dover Co., New York)

P. J. Hilton. *An introduction to homotopy theory.* Pp. viii, 142. 15s. 1953. Cambridge tracts, 43. (Cambridge University Press)

V. Hlavaty. *Differential line geometry.* Translated by H. Levy. Pp. x, 495. Fl. 22.50; cloth, fl.25. 1953. (Noordhoff, Groningen)

A. S. Householder. *Principles of numerical analysis.* Pp. x, 274. 48s. 1953. (McGraw-Hill)

N. Jacobson. *Lectures in abstract algebra. II. Linear algebra.* Pp. xii, 280. 42s. 1953. (Van Nostrand, New York; Macmillan, London)

G. Julia. *Cours de géométrie infinitésimale. I.* 2nd edition. Pp. xv, 102. 2000 fr. 1953. (Gauthier-Villars, Paris)

M. G. Kendall. *Exercises in theoretical statistics.* Pp. vii, 179. 20s. 1954. (Griffin)

E. Landau. *Handbuch der Lehre von der Verteilung der Primzahlen.* Reprinted, with appendix by P. T. Bateman. Pp. 1001. Two volumes. $17.50. 1953. (Chelsea Co., New York)

W. Lietzmann. *Der Pythagoreische Lehrsatz.* 7th edition. Pp. 96. DM 3.60. *Wo steckt der Fehler?* 3rd edition. Pp. 185. DM 5.80. 1953. (Teubner, Stuttgart)

D. V. Lindley and J. C. P. Miller. *Cambridge elementary statistical tables.* Pp. 35. 5s. 1953. (Cambridge University Press)

J. E. Littlewood. *A mathematician's miscellany.* Pp. 136. 15s. 1953. (Methuen)

N. W. McLachlan. *Complex variable theory and transform calculus.* 2nd edition. Pp. xi, 388. 55s. 1953. (Cambridge University Press)

E. J. McShane. *Order-preserving maps and integration processes.* Pp. vi, 136. 22s. 1953. *Annals of Mathematics* studies, 31. (Princeton University Press; Geoffrey Cumberlege, London)

K. Menger. *Géométrie générale.* Pp. 80. 1000 fr. 1954. Mémorial des sciences mathématiques, 124. (Gauthier-Villars, Paris)

H. Milloux. *Traité de Théorie des Fonctions. I, fasc. 1. Principes. Méthodes générales.* Pp. viii, 300. 4500 fr. 1953. (Gauthier-Villars, Paris)

P. M. Morse and H. Feshbach. *Methods of theoretical physics. I. II.* Pp. xxii, 1978. £6 each volume. 1953. (McGraw-Hill)

R. Rothe. *Höhere mathematik. II.* 9th edition, edited by W. Schmeidler. Pp. 210. DM 7.60. 1953. *Höhere mathematik. III.* 6th edition, edited by W. Schmeidler. Pp. 242. DM 8.20. 1953. *Hohere mathematik. VI.* By I. Szabó. Pp. 251. DM 17.60. 1953. Mathematische Leitfäden, 22, 23, 45. (Teubner, Stuttgart)

W. M. Smart. *Celestial mechanics.* Pp. vii, 381. 70s. 1953. (Longmans)

D. Smeltzer. *Man and number.* Pp. viii, 114. 7s. 6d. 1953. (Black)

C. Stern. *Children discover arithmetic.* Pp. xxiv, 295. 25s. 1953. (Harrap)

R. W. Stott. *Essential calculus.* Pp. 80. 3s. 3d. 1953. (University of London Press)

J. A. Tampkins. *Commercial mathematics. I. II.* Pp. 123, 122. Limp, 5s. each; boards, 6s. each. 1953. (English Universities Press)

A. Tarski. *Undecidable theories.* Pp. xi, 98. 18s. 1953. (North-Holland Publishing Co., Amsterdam)

A. C. Zaanen. *Linear analysis.* Pp. vii, 600. 76s. 1953. Bibliotheca Mathematica, 2. (North-Holland Publishing Co., Amsterdam; Noordhoff, Groningen)

Contributions to the theory of Riemann surfaces. Edited by L. Ahlfors. Pp. 264. 25s. 1953. *Annals of Mathematics* studies, 30. (Princeton University Press; Geoffrey Cumberlege, London)

Graphs of the Comptom Energy-angle relationship and the Klein-Nishina formula from 10 *Kev to* 500 *Mev.* Pp. 88. 55c. 1953. (National Bureau of Standards, Washington)

Modern developments in fluid dynamics. High speed flow. I. II. Edited by L. Howarth. Pp. xvi, 475; viii, 477-875. 84s. 1953. (Geoffrey Cumberlege, Oxford University Press)

Simultaneous linear equations and the determination of eigenvalues. Pp. 126. $1.50. Applied mathematics series, 29. *Tables of natural logarithms for arguments between zero and five to sixteen decimal places.* Pp. 501. $3.25. Applied mathematics series, 31 (revison of No. 10). 1953. (National Bureau of Standards, Washington).

A SCHOOLBOY'S LETTER, 1732

The Hon. T. Lennard Barrett writes to his aunt from Harrow. *Original spelling retained.* Barrett Lennard Archives (D/DL C43/3).

Dear Aunt, March the 29, 1732

As you have always been exceeding good to me, for which I shall never be able to make you a Return, I beg you'd now please to grant me this Request viz. to Leave off Learning Mathematicks, this Dear Madam may att first seem an unreasonable thing, but I hope the Reasons which I am now going to give you will make you, according to your usual goodness, grant me this Request.

You Very well know, Dear Madam, that I have been an old Border, & therefore ought not to be slighted by Weston, which I have been very much. For This Morning I heard that Mr Weston was going out with some off the Gentlemen to teach them to measure Ground ; upon this Report I took my hat and stood among the Rest off those who were going, when to my Great Surprise Mr Weston came down Stairs and bid me begon, for I only wanted to be Idle & told me I shou'd not go with him. This has netled me very much because the Boys he took out with him were no farther advanc'd in Mathematick's than I, As for his saying I only wanted to be Idle, I hope you will believe me when I tell you upon my Honour that the Chief and only end of my Desire to go out a measuring was to be instructed in that Art. Neither do I think I ever gave him half so much Reason to think me prone to Idleness as Barnet and Philips who went with him. If You will be so good then, Dear Madam, to grant me my earnest Request of Leaving off Mathematicks, I shall be exceedingly oblig'd to you.

You may Rember, Dear Aunt, that I have often told you that Mr Evans taught me Mathematicks in private, I shall still continue to Learn Mathematicks of Mr Evans who is more able to teach me than Weston. For I can asure you I have learnt more from Mr Evans than I ever did from Weston. I hope I have now given you Sufficient Reasons for my Desireing to Leave Learning Mathematicks from Mr Weston ; I therefore beg you'd be so good to me (Dear Madam) as to Grant my Desire. There is an old Saying one Story is good till another is told, but I can assure you on my Honour that what I have told you is true. I beg therefore that if you grant me my Request, that you will not be mov'd by Weston's fawning and funning, who cares no more for you nor I than what money he can make off us, else he woul'd not have us'd you in the manner he did when you came from Lady Lennards, to give you old heartychoaks for supper & to Lay you in Bed with frowsy Bet Rosam, to be devoured by Buggs.

The Quarter is now begining, therefore tis a very fit time to Leave off Mathematicks. I beg you'd Let me have a Letter from you as soon as possible, in which I hope to recieve Orders to tell Mr Weston that I Don't Learn Mathematicks any Longer off him, which will be an inexpressible pleasure to,

Dear Aunt,

Your most Dutifull Nephew,

T. Lennard Barrett.

P.S. I shall take Double pains in Latin and everything else. I shall be so far from losing the Mathematicks I have already got that I don't doubt to make great improvements in them under Mr Evans' Care. Pray Don't tell Weston I Learn of Mr Evans.

[Per Dr. B. E. Lawrence.]

REPORT OF THE MEETING OF THE TEACHING COMMITTEE

6TH JANUARY, 1954

The new Teaching Committee (one is appointed every four years at the A.G.M. and must contain at least 15 new members) met at King's College, London. 34 members were present. The names of the Committee are given below, with the sub-committees to which they were elected at this meeting.

The Chairman referred to the loss they had sustained by the death in December of Miss W. Garner, who had been the first secretary of the Manchester Branch, and an ever active and helpful member both of the Branch and of the Teaching Committee.

The Committee received with regret the news that owing to pressure of other work Mr. M. W. Brown was not available for re-election as Hon. Secretary, and recorded their thanks to him for all his help during the last two years. Mr. B. J. F. Dorrington was elected to succeed him. In the interests of continuity the present Chairman offered himself for re-election, and there was no other nomination.

The revised draft reports on the teaching of mathematics in Primary Schools and in Technical Colleges were both left with their respective sub-committees for final shaping and publication. In view of some things said at the Annual Meeting, it may be well to state here that the Teaching Committee does not hold that what is taught in schools other than Grammar Schools, under the title of Mathematics, has nothing to do with the mathematics in the Grammar Schools. One of the main objects of at least three of the present sub-committees is to relate the teaching in the different kinds of schools. The non-mathematician who is called upon to teach mathematics should be enabled to connect himself with the stream of culture that we know as mathematics. The pupil should be similarly, no matter how distantly, related to it, and not deprived of such mathematical development as may be within his capacity even though that capacity may be very small.

With this in mind, the report on the teaching of mathematics in Modern Schools is also being pushed forward, and another sub-committee is continuing to try to elucidate precisely those relationships which are believed to exist between the mathematics in the various types of school. Sub-committees on the teaching of Algebra and Analysis in sixth forms are also making good progress, while another is dealing with " History of Mathematics " as an aid to the teaching of the whole subject and also as an unwelcome intruder into the examination system.

Arrangements have been made for keeping up to date the List of Books suitable for school libraries, and the Report on the use of visual methods,* both of which were circulated to all members of the Association in February.

A memorandum from one of our members on some proposals for changes in the Common Examination for entrance to Public Schools is receiving attention.

When the sub-committee on mathematics in Technical Colleges has finished its report it is to be asked to turn its attention to the need for investigating sources of mathematics in other sciences.

It may not be out of place here—although the matter is not an act of the Teaching Committee's—to record that, by invitation of the Federation of British Industries, the Chairman of the T.C. and the past Chairman (Mr. W. J. Langford) were present at a conference convened on 14th January by the F.B.I. to discuss the shortage of mathematicians and other scientists in industry, in the schools and in the universities. J. T. C.

* The Association for Multi-Sensory Aids, mentioned early in the Report, is now the Association for Teaching Aids in Mathematics, and the Secretary, Mr. R. H. Collins, is now at 97 Chequer Road, Doncaster.

THE TEACHING COMMITTEE 1954–1958

Ex officio

The President	
The Hon. Treasurer	Mr. J. B. Morgan, Harrow School.
The Hon. Secretaries	Mr. F. W. Kellaway, (*d*), North Herts. Technical College. Miss W. A. Cooke, High School, Slough.
The Editor	Prof. T. A. A. Broadbent, (*h*, *s*), Royal Naval College, Greenwich.

Universities

Busbridge, Dr. I. W., (*f*)	St. Hugh's College, Oxford.
Cartwright, Dr. M. L., (*f*)	Girton College, Cambridge.
Combridge, Mr. J. T., (*s*)	King's College, London.
Cooper, Prof. J. L. B., (*h*)	University College, Cardiff.
Goodstein, Prof. R. L.	University College, Leicester.
Hamill, Dr. C. M.	University of Sheffield.
Maxwell, Dr. E. A.	Queens' College, Cambridge.
Newman, Prof. M. H. A., (*l*)	University of Manchester.

Training Colleges and Departments of Education.

Cripwell, Mr. R. H., (*c*)	Didsbury Training College, Manchester.
Daltry, Mr. C. T., (*g*, *h*)	University of London Institute of Education.
Sowden, Miss K. M., (*b*)	City of Bath Training College.
Vesselo, Mr. I. R., (*b*, *d*)	County Training College, Alsager, Stoke-on-Trent.
Williams, Mrs. E. M., (*b*, *c**, *g*, *s*)	Whitelands Training College, S. W. 15.

Technical Colleges and Schools.

Avery, Mr. A. J. L., (*d*)	Derby Technical College.
Chybalski, Mrs. I. P., (*d*)	S. E. Essex Technical College.
Lowry, Mr. H. V., (*d*)	Woolwich Polytechnic.
Phillips, Mr. F. J., (*d*)	Battersea Polytechnic.

Secondary Schools

Armistead, Mr. W., (*g*)	Christ's Hospital.
Barton, Mr. A., (*p*)	Cheltenham College.
Bromby, Miss H.	Southampton Girls' Grammar School.
Brown, Mr. M. W., (*b*, *g**, *s*)	Holloway School.
Cawley, Miss J. M., (*h*)	Whalley Range High School for Girls, Manchester.
Dorrington, Mr. B. J. F., (*b*, *s*)	Downham Secondary School.
Giuseppi, Miss Y. B., (*b*)	West Norwood Secondary School.
Green, Mr. R. E.	City of London School.
Holman, Miss E. M., (*l**)	Manchester High School for Girls.
Imeson, Mr. K. R., (*b*, *g*)	Nottingham High School.
Langford, Mr. W. J., (*f**, *h*, *l*)	Battersea Grammar School.
Penfold, Mr. A. P., (*f*)	Battersea Grammar School.
Prag, Mr. A., (*h**)	Westminster School.
Quadling, Mr. D. A., (*p*)	Marlborough College.
Snell, Mr. K. S., (*p**)	Harrow School.
Walker, Mr. R.	Stowe School.

Preparatory and Primary School Interests

Adams, Miss L. D., (*b*, *c*)	formerly Ministry of Education Inspectorate.
James, Mr. E. J., (*b*)	Redland College, Bristol.
Morris, Miss R. E., (*c*)	Dinorben School, Wallington.
Sutcliffe, Miss E. W., (*c*)	The Maynard School, Exeter.
Talbot, Mr. B. L., (*p*)	St. Peter's School, Seaford.
Theakston, Mr. T. R., (*g*)	City of Coventry Training College.

Special Interests

Riley, Mr. A. W., (*b**, *c*)	Wolverhampton L.E.A. Inspectorate
Robson, Mr. A., (*l*)	formerly Marlborough College.
Rollett, Mr. A. P., (*g*)	Ministry of Education Inspectorate.
Tuckey, Mr. C. O., (*f*)	formerly Charterhouse.

Sub-committees

b	Secondary Modern Schools.	*h*	History of Mathematics.
c	Primary Schools.	*l*	Sixth-form Algebra.
d	Technical Colleges.	*p*	Preparatory Schools.
f	Sixth-form Analysis.	*s*	Standing Sub-committee.
g	Professional Training.	*	Convener or Secretary.

CORRESPONDENCE

THE HIGHER GEOMETRY REPORT

Sir,—I left the discussion on this report very dissatisfied and conversation afterwards indicated that I was not alone in this. That the discussion virtually died and had to be rescued forcibly perhaps confirms this impression. If I am right, may I boldly suggest reasons for this failure which seem worth recording.

First, I think many members present felt the concentration on abstract geometry did not concern them at all. It is small comfort to genuine inquirers on three-dimensional (pedestrian) geometry to be told " Teach them abstract n-dimensional geometry, and then just put $n=3$ ".

Secondly, many of us were suspicious of the actual content of abstract geometry " quite different from physics " and are convinced that this is work for the universities, who frequently and piously decry premature specialization. Personally as a boy I delighted in a spatial course as in Mr. Durell's *Modern Geometry* and *Projective Geometry* and am cave-man enough to think so still. Yet I found the change of ideas at Cambridge not too great, though the course generally duller! Further, many grammar school pupils who later specialize in mathematics do little or no projective or complex geometry at all.

Finally, when questions are set in scholarship papers on abstract geometry, it would make the preponderance of the few interested schools more marked, a step I should deplore.

Yours, etc., H. Ivor Jones.

Sir,—The presidential address and the discussion of the geometry report at the recent annual meeting have encouraged me to put forward some ideas which arise out of my work at a Rudolf Steiner school.

My senior colleagues have now for 25 years taught projective geometry to pupils of varying intellectual ability and I have been fortunate enough to be able to join in this work for the last seven years. It may be that some of the experience gained, although not directly applicable to the grammar schools, may help eventually in the solution of some of the problems raised in the discussion.

It seemed to me that some of the members, while wholeheartedly welcoming the report as a guiding line for the work with their ablest pupils, were somewhat worried on two counts. They seemed to wonder how this work would stand in relation to geometry teaching earlier in the school and a note of disappointment could be detected that there was no indication how the more systematic treatment of essentially euclidean geometry could be approached now that the strictly axiomatic development has been discarded. The latter seems important because there is a danger that some pupils never make a proper acquaintance with a coherent, logical and deductive edifice of thought.

It seems to me that both these difficulties can be met if " pure " projective geometry finds a place in the school curriculum somewhere at 5th form level.

In the general course of mathematics teaching, one usually goes from the concrete to the abstract and very frequently one follows the course of historic development. The traditional geometry course, for instance, starts with Euclid which forms both historically and logically the basis for work with rectangular cartesian co-ordinates. On these principles alone, the introduction of algebraic projective geometry at 6th form level calls for the teaching of " pure " projective geometry earlier on. I am certain that such concepts as " ideal elements " would have much firmer roots in a pupil's mind if they were first experienced as limiting positions of concrete elements on the drawing board. " Involution " would lose much of its enigmatic quality if actual involution ranges had been constructed ; perhaps on a fixed side of an otherwise variable self polar triangle with respect to a circle. The needs of the future specialist might thereby be met.

I think, however, that this approach would also be of great pedagogical value to the rest of the class, that is, to the vast majority.

The propositions of incidence of points, lines and planes in space can be established quite intuitively. This is in fact an exercise which trains the children's powers of visual imagination, the development of which is often sadly neglected. These propositions can be developed in such a way that then they exhibit most clearly the principle of duality, one of the most important and most beautiful facts of geometry. Moreover, their immediate consequences such as Desargues' triangle theorem and the properties of quadrangles and quadrilaterals are both interesting and surprising. This can also be used as the basis for drawing exercises which can give beautiful results and which require great care and accuracy. The introduction of points and lines at infinity can, of course, cause difficulties. We do, however, find that they can be met by slowly accustoming the children to these ideas.

Perhaps it would be helpful if I gave a rough outline of the geometry curriculum in use at Michael Hall, where I am teaching. Geometry starts seriously when the children are about 11 years old and by the time they are 14 they have been introduced to the main metrical properties of triangles, quadrilaterals and circles. This is done by a combination of experiment and deduction from intuitively obvious facts with great emphasis on drawing. The development in the next three years is then planned to culminate in projective geometry. Here teachers vary in their method. I myself have used quite a number of different approaches. The most successful is perhaps likely to be as follows :

1st year : Regular polygons and polyhedra giving a chance to revise much of the previous work followed by an introduction to descriptive geometry.

2nd year : Conic sections treated as loci of various kinds. Here the transformation of one form into another is stressed and there is, therefore, a good opportunity to accustom the pupils to points moving through infinity. Concurrently with this, one would take descrip-

tive geometry up to sections of solids by inclined planes, linking up with the work on conics.

3rd year : Projective Geometry : Propositions of Incidence in space, Desargues' Theorem, Quadrangle and Quadrilateral, Conics as products of projective ranges and pencils, Pascal's and Brianchon's theorems leading back to the quadrilateral and pole and polar via degenerate hexagons, Cross ratio, using similar triangles to link up with the previous work.

I have myself never actually done the work quite in that order, but this is how I should do it next time. It must of course be said that we have two very great advantages at Michael Hall which enable us to pursue such a course. First of all, we need not prepare our pupils for external examinations until they are between 17 and 18 and secondly, our timetable arrangements make really concentrated work possible. When it comes to examinations, however, the preparation for the geometry required at Ordinary Level does not take more than a term's work.

Personally, I was delighted with the geometry report. I should, therefore, be very glad if my suggestion to introduce projective geometry earlier in the school curriculum were considered to be in line with the spirit of that report. I hope also that it is in line with the plea from the presidential chair for a widening of horizons in mathematics teaching. I cannot help feeling that it is high time that a larger public should become acquainted with modern thought in the realm of Mathematics. The general conception of mathematics as a complete subject incapable of further development might then disappear and a conception of space might be developed which would make it easier to follow the development of science.

Yours, etc., H. Gebert.

COMBINATORIAL NOTATION

To the Editor of the *Mathematical Gazette*.

Sir,—May I enquire of your readers their opinions on a suggested new mathematical notation? I refer to the use of $(a:b:c:d:e)$ to denote the number of ways of distributing $(a+b+c+d+e)$ objects among boxes labelled Box 1, Box 2, . . . Box 5, so that there are a objects in Box 1, b in Box 2, . . . , e in Box 5. And similarly for any larger or smaller number of boxes.

My original reason for adopting this notation was the ease of typing $(a:b)$ as compared with current alternatives for binomial coefficients. But subsequent experience in teaching the theory of combinations for probability purposes made me think it simplifies the presentation of this theory. One reason is, that the notation helps to emphasise the symmetry between the boxes.

To develop the theory, we consider first two boxes. Since their labels can be interchanged, we have

$$(a:b)=(b:a) \quad \ldots\ldots\ldots\ldots (1)$$

while if all objects are to go into Box 1, this can be done in only one way, so that

$$(a:0)=1 \quad \ldots\ldots\ldots\ldots (2)$$

If there are a objects, and only one is to go into Box 2, this one can be chosen in a ways, so that

$$(a-1:1)=a, \quad \ldots\ldots\ldots\ldots (3)$$

Next, we observe that, to obtain $(a:b)$, we can take $(a+b-1)$ objects and

put a into Box 1, and $(b-1)$ into Box 2, and then add a last object to Box 2, or we can put $(a-1)$ into Box 1, and b into Box 2, and add the last object to Box 1. Thus

$$(a:b)=(a:b-1)+(a-1:b) \quad \text{.............................(4)}$$

Equivalently, we can classify all the $(a:b)$ ways into those in which a particular object occurs in Box 1, and those in which this object occurs in Box 2.

We can use (1), (2), and (4) to develop Pascal's triangle, and to prove the binomial theorem, in the easily remembered form

$$(a+b)^n=(n:0)a^nb^0+(n-1:1)a^{n-1}b^1+\dots+(n-r:r)a^{n-r}b^r+\dots+(0:n)a^0b^n.$$

To develop the theory further, we observe that $(a:b:c:d)$ can be obtained by first deciding which objects are to go into Boxes 1 and 2, and which into Boxes 3 and 4, giving $(a+b:c+d)$ ways. Then there are $(a:b)$ ways of distributing the objects between Boxes 1 and 2, and $(c:d)$ between Boxes 3 and 4. Hence

$$(a:b:c:d)=(a+b:c+d)(a:b)(c:d) \quad \text{......................(5)}$$

Now by relabelling the boxes, $(a:b:c:d)=(a:c:b:d)$, so that

$$(a+b:c+d)(a:b)(c:d)=(a+c:b+d)(a:c)(b:d) \quad \text{.....................(6)}$$

The identities (1)–(4) and (6) form a convenient basis for the mathematical theory. Replacing a by $(a-1)$ in (6), and putting $b=1$, $d=0$, we have

$$(a:c)(a-1:1)(c:0)=(a-1+c:1)(a-1:c)(1:0)$$

so that, using (2) and (3),

$$\begin{aligned}(a:c)&=\frac{(a+c)}{a}(a-1:c)\\ &=\frac{(a+c)(a+c-1)\dots(a-(a-1)+c)}{a\quad(a-1)\quad\dots\quad 1}(0:c)\\ &=\frac{(a+c)(a+c-1)\dots(c+1)}{a(a-1)\dots 3\,.\,2\,.\,1} \quad \text{..................................(7)}\end{aligned}$$

the usual expression.

As an example of a simplified approach suggested by the notation, we will consider the probability that, with well-shuffled cards, at Bridge, no player obtains as many as 5 cards of a suit.

There are $(13:13:13:13)$ dealings altogether (taking the players as Boxes 1, 2, 3, 4). Now all the suits must be distributed as $(4:4:4:3)$, or some rearrangement of this, and each player must have just one "short" suit. There are 4! ways of distributing the short suits among the four players Hence the probability is

$$4!(3:4:4:4)(4:3:4:4)(4:4:3:4)(4:4:4:3)/(13:13:13:13)$$

which, using an easy consequence of (6) and (7), to wit

$$\log(a:b:c:d)=\log(\Sigma a)!-\Sigma\log a!$$

can be found to be about 1 in 70 millions. Improbable as this is, it is more than 10^{19} times as probable as a deal in which each player has a complete suit, which has probability

$$4!/(13:13:13:13),$$

Yours truly, G. A. Barnard.

Imperial College,
London

ANNUAL MEETING OF THE MATHEMATICAL ASSOCIATION 1955

The Annual Meeting of the Association for 1955 will be held at Leicester, April 13–16. Accommodation has been made available by the courtesy of University College, Leicester. Further details of the programme of papers and other activities will be sent to members during December, 1954.

FOR SALE

(i) A complete run of the *Mathematical Gazette* from 1910 to 1944, unbound, in good condition. Offers to Mrs. G. Beaven, Perrymead, South Widcombe, East Harptree, near Bristol.

(ii) All copies of the *Mathematical Gazette*, 1944–1953 inclusive (Nos. 278–322), with index parts ready for binding. Offers to L. A. W. Jones, Berkhamsted School, Herts.

ASSOCIATION FOR TEACHING AIDS IN MATHEMATICS

The first meeting of this Association in the North of England is to be held on October 16, at the Education Centre, Manchester University, Dover Street, Manchester, from 10.30 a.m. to 4 p.m. The meeting is open to all, whether members of the A.T.A.M. or not ; a copy of the detailed programme, which will consist of demonstrations, showing of films and filmstrips, exhibition of models, may be obtained by sending a stamped addressed envelope to the Secretary, Mr. C. Birtwistle, 1 Meredith Street, Nelson, Lancs.

BUREAU FOR THE SOLUTION OF PROBLEMS

Enquiries should be addressed to Dr. G. A. Garreau, 90 Wyatt Park Road, Streatham Hill, London, S.W. 2, accompanied by a stamped addressed envelope. Applicants should state the source of their problems, and the names and authors of textbooks on the subject, to which they can refer. Whenever questions from the Cambridge Mathematical Scholarship volumes are sent, it is not necessary to copy the question but only to send the reference, *i.e.* volume, page and number. This applies to Newnham and Girton papers of 1952 and after. If, however, the questions are taken from the papers in Mathematics set to Science candidates, these should be given in full. The names of applicants will not be published.

All solutions must be returned to the Secretary of the Bureau.

THE CARDIFF BRANCH

Officers appointed for the 1953–54 session were :

Mr. A. H. Pope, *President* ; Mr. R. A. Jones, *Treasurer* ; Mr. W. H. Williams, *Secretary*.

9th November, 1953. The Branch was very pleased to hear an address from Professor T. A. A. Broadbent, the President of the parent Association for 1953, on " The Principles of Coordinate Geometry ". Professor Broadbent made a plea for the unification of method in the treatment of the subject, and he developed a case for making the subject rest upon two main ideas which he described as " distance-direction " and " distance-ratio " principles.

30th November, 1953. Mr. I. E. Hughes, H.M.I., addressed the Branch on " The Place of Mathematics in Present-day Education ". Mr. Hughes dealt briefly with the change in teaching methods in recent years at the infant and junior levels of primary education, and then considered in more detail the place of mathematics in secondary education.

1st February, 1954. Dr. A. J. C. Wilson gave an address on " Probability

Distributions in X-Ray Crystallography ". When a beam of X-Rays falls on a small crystal which is rotated about an axis perpendicular to the incident beam, the rays fall successively on different crystal planes and are reflected to form a pattern on a film placed perpendicular to the original beam. The film indicates the intensity of the scattered X-Ray beams in the different directions. From theoretical considerations Dr. Wilson developed formulae for the probable distribution of intensity of the scattered beams, and illustrated with slides the comparison between predicted values and those obtained for actual crystals.

15th March, 1954. Dr. Kemp, statistician to British Nylon Spinners, gave an address on " Statistics in Industry, with special reference to the Nylon Industry ". He explained that the statistician in this industry had two main functions : he had to give advice on the control of the quality of the manufactured yarn, and he had to design suitable experiments to test various properties of the yarn. Dr. Kemp dealt in detail with both these aspects.

W. H. WILLIAMS, *Hon. Secretary.*

EUREKA

Everyone who is interested in mathematical topics will be interested in *Eureka,* the annual journal of the Archimedeans, the Cambridge University undergraduate mathematical society. The articles in *Eureka* are concerned with interesting and amusing facets of mathematics (and other things), and there are many problems and puzzles especially designed to worry *Eureka's* readers in their otherwise leisure moments. *Eureka* is a journal designed for those who teach, those who learn, and those who enjoy, mathematics.

Copies can be obtained from the Business Manager, *Eureka,* The Arts School, Bene't Street, Cambridge ; the usual price is 2s. per copy. Subscriptions can be made under one of three systems : (i) standing orders may be placed for copies to be sent as published, at the rate of 2s. 2d. per copy, payable on receipt of each edition ; (ii) for a subscription of 10s. or more, copies will be sent at the rate of 1s. 8d. each until the payment is exhausted ; (iii) orders for ten copies or more of the same edition will be sent, post free, at the rate of 1s. 6d. per copy.

ARCHIV DER MATHEMATIK

Volume V of the *Archiv der Mathematik* appears as a recognition of Professor A. Ostrowski's sixtieth birthday on September 25, 1953. The volume will be published in two parts, of 250 pages each, during 1954, and the price of the volume is 72 Swiss francs. About 70 papers by Ostrowski's friends, colleagues and pupils will form the contents of this special volume, which is published by Messrs. Birkhäuser, Basel 10, Switzerland.

BOOKS FOR REVIEW

E. N. da C. Andrade. *Sir Isaac Newton.* Pp. 140. 7s. 6d. 1954. Brief Lives, 11. (Collins)

A. Blanc-Lapierre et **R. Fortet.** *Théorie des fonctions aléatoires.* Pp. xvi, 693. 6500 fr. 1953. (Masson, Paris)

P. F. Byrd and **M. Friedman.** *Handbook of elliptic integrals for engineers and physicists.* Pp. xiii, 355. DM 36 ; bound, DM 39.60. 1954. Grundlehren der mathematischen Wissenschaft, 67. (Springer, Berlin)

C. Carathéodory. *Theory of functions of a complex variable. I.* Translated by F. Steinhardt. Pp. xii, 301. $5. 1954. (Chelsea Co., New York)

T. W. Chaundy, P. R. Barrett and C. Batey. *The printing of mathematics.* Pp. ix, 105. 15s. 1954. (Geoffrey Cumberlege, Oxford University Press)

I. M. Copi. *Symbolic logic.* Pp. xiii, 355. \$5. 1954. (Macmillan, New York)

J. P. Dalton. *Symbolic operators.* Pp. xvi, 194. 30s. 1954. (Witwatersrand University Press)

A. Denjoy. *L'énumeration transfinie. III. Études complémentaires sur l'ordination* Pp. 615–772. 2400 fr. *IV. Notes sur les sujets controversés.* Pp. 773–969. 3200 fr. 1954. (Gauthier-Villars, Paris)

V. C. A. Ferraro. *Electromagnetic theory.* Pp. viii, 555. 42s. 1954. (Athlone Press. University of London)

H. C. Fryer. *Elements of statistics.* Pp. viii, 262. 38s. 1954. (John Wiley, New York; Chapman & Hall, London)

R. Garnier. *Cours de cinématique. I.* 3rd edition. Pp. ix, 244. 4000 fr. 1954. (Gauthier-Villars, Paris)

M. H. Hansen, W. N. Hurwitz and W. G. Madow. *Sample survey methods and theory. I. Methods and applications.* Pp. xxii, 638. 64s. *II. Theory.* Pp. xiii, 332. 56s. 1953. (John Wiley, New York; Chapman and Hall, London)

G. E. Hay. *Vector and tensor analysis.* Pp. viii, 193. \$1.50; cloth, \$2.75. 1954. (Dover Publications, New York)

R. V. Heath. *Mathemagic.* Pp. 126. \$1. 1953. (Dover Publications, New York)

W. V. D. Hodge and **D. Pedoe.** *Methods of algebraic geometry. III.* Pp. x, 336. 40s. 1954. (Cambridge University Press)

O. D. Kellogg. *Foundations of potential theory.* (Rep.) Pp. ix, 384. \$1.90; cloth, \$3.95. 1954. (Dover Publications, New York)

L. M. Kells. *Elementary differential equations.* 4th edition. Pp. x, 266. \$4. 1954. (McGraw-Hill)

R. E. Langer. *A first course in ordinary differential equations.* Pp. xii, 249. 36s. 1954. (John Wiley, New York; Chapman and Hall, London)

G. G. Lorentz. *Bernstein polynomials.* Pp. x, 130. 45s. 1953. Mathematical expositions, 8. (Toronto University Press; Geoffrey Cumberlege, London)

C. C. MacDuffee. *Theory of equations.* Pp. vii, 120. 30s. 1954. John Wiley, New York; Chapman and Hall, London)

E. Mach. *The principles of physical optics.* (Rep.) Pp. x, 324. \$1.75; cloth, \$3.50. 1954. (Dover Publications, New York)

E. A. Maxwell. *An analytical calculus.* I, II. Pp. xii, 165; vi, 272. 15s.; 18s. 1954. (Cambridge University Press)

N. I. Muskhelishvili. *Singular integral equations.* Translated from the second Russian edition by J. R. M. Radok. Pp. 447. Fl. 28.50. 1953. (Noordhoff, Groningen)

N. I. Muskhelishveli. *Some basic problems of the mathematical theory of elasticity.* Translated from the third Russian edition by J. R. M. Radok. Pp. xxxi, 704. Fl. 38. 1953. (Noordhoff, Groningen)

H. F. P. Purday. *Linear equations in applied mechanics.* Pp. xiv, 240. 20s. 1954. (Oliver & Boyd)

W. van O. Quine. *From a logical point of view.* Pp. vi, 184. 22s. 6d. 1953. (Harvard University Press; Geoffrey Cumberlege, London)

H. Reichenbach. *Nomological statements and admissible operations.* Pp. 140. 27s. 1954. (North-Holland Publishing Co., Amsterdam)

K. Reidemeister. *Die Unsachlichkeit in der Existenzphilosophie.* Pp. iv, 40. DM. 4.80. 1954. *Geist und Wirklichkeit.* Pp. iii, 92. DM. 8.60. 1953. (Springer, Berlin)

W. Haydn Richards. *Arithmetic made easy. II,* 1, 2. Pp. 96; 96. 3s. each. 1954. (Harrap)

M. G. Salvadori. *The mathematical solution of engineering problems.* Pp. x, 245. 34s. 1954. (Columbia University Press; Geoffrey Cumberlege, London)

W. Sierpinski. *On the congruence of sets and their equivalence by finite decomposition.* Pp. 117. N.p. 1954. Lucknow University Studies, 20. (Lucknow University)

QUEENSLAND BRANCH

Report for the Year 1953–1954

This is the thirty-second annual report of this Branch. We have to record with regret, the untimely and unexpected death, in August 1953, of Mr. S. G. Brown, M.A., B.Sc., who was one of the foundation members of the Branch and who, except for a short period of absence from Queensland, was one of its most active members.

The 1953 Annual Meeting was held at the University, George Street, on 23 May, 1953. The Annual Report and the Statement of Receipts and Expenses were submitted to the meeting and were adopted. After the election of officers for the year, Mr. M. P. O'Donnell read a paper entitled " Solving equations ".

Two other general meetings were held during the year, both at the University. At the first, on 7 August, 1953, Associate Professor J. P. McCarthy read a paper on " Modern Civilisation's debt to Mathematics ", and at the second, on 30 October, 1953, Mr. S. E. Reilly gave an interesting lecture on " Position finding ".

The Statement of Receipts and Expenses for the year shows a credit balance of £12 4s. 5d. A great part of the expenses is incurred in postal charges in connection with the circulation of the *Gazette* amongst Associate Members. During the year the *Report on the Teaching of Higher Geometry* came to hand and is in course of circulation. The number of members of the Branch is 36, which includes two life members and 11 members of the M.A. The attendance at meetings has been quite good.

The present committee is :

President : Professor E. F. Simonds ; *Vice-Presidents :* Mr. E. W. Jones, Mr. H. M. Finucan ; *Hon. Secretary and Hon. Treasurer :* Associate Professor J. P. McCarthy ; *Members :* Miss E. H. Raybould, Miss M. A. Popple, Mr. A. W. Young, Mr. P. B. McGovern, Mr. I. A. Evans.

J. P. McCarthy
Hon. Secretary

BOOKS FOR REVIEW

W. Ackerman. *Solvable cases of the decision problem.* Pp. viii, 114. 24s. 1954. (North-Holland Publishing Co.)

D. J. Aitken and K. B. Henderson. *Algebra. Its big ideas and basic skills. I.* Pp. xi, 419. 22s. 1954. (McGraw-Hill)

D. N. de G. Allen. *Relaxation methods.* Pp. ix, 203. 36s. 1954. (McGraw-Hill)

G. Aumann. *Reelle Funktionen.* Pp. viii, 416. DM 56 ; cloth, DM 59.60. 1954. Grundlehren der mathematischen Wissenschaften, 68. (Springer, Berlin)

J. H. Avery and M. Nelkon. *An introduction to the mathematics of physics.* Pp. vii, 178. 9s. 6d. 1954. (Heinemann)

R. A. Beaumont and R. W. Ball. *Introduction to modern algebra and matrix theory.* Pp. xii, 331. 30s. 1954. (Constable)

O. Becker. *Grundlagen der Mathematik in geschichtlicher Entwicklung.* Pp. xii, 424. DM 26. 1954. (Alber, Freiburg)

D. Blackwell and M. A. Girshick. *Theory of games and statistical decisions.* Pp. xi, 355. 60s. 1954. (John Wiley, New York ; Chapman and Hall)

W. Blaschke. *Analytische Geometrie.* 2nd edition. Pp. 190 Sw. fr. 19.60. 1954. *Projective Geometrie.* 3rd edition. Pp. 197. Sw. fr. 19.60. 1954. Mathematische Reihe 16, 17. (Birkhäuser, Basel)

G. Bouligand. *Mécanique rationnelle.* 5th edition. Pp. xxxii, 568. 2400 fr. 1954. (Vuibert, Paris)

W. Briggs and G. H. Bryan. *The tutorial algebra. I.* 6th edition, revised and rewritten by G. Walker. Pp. xii, 491. 1954. (University Tutorial Press)

H. A. Buchdahl. *Optical aberration coefficients.* Pp. xx, 336. 50s. 1954. (Geoffrey Cumberlege, Oxford University Press)

L. H. Clarke. *Fun with figures.* Pp. vii, 87. 6s. 1954. (Heinemann)

A. Denjoy. *Mémoire sur la dérivation et son calcul inverse.* Pp. vii, 380. 2700 fr. 1954. (Gauthier-Villars, Paris)

F. I. Frankl and E. A. Karpovich. *Gas dynamics of thin bodies.* Translated by M. D. Friedman. Pp. viii, 175. 38s. 1954. (Interscience Publishers)

B. Fruchter. *Introduction to factor analysis.* Pp. xii, 280. 27s. 6d. 1954. (Van Nostrand, New York ; Macmillan)

B. V. Gnedenko and A. N. Kolmogorov. *Limit distributions for sums of independent random variables.* Translated by K. L. Chung. Pp. ix, 264. $7.50. 1954. (Addison-Wesley, Cambridge, Mass.)

C. Goffman. *Real functions.* Pp. xii, 263. 30s. 1954. (Constable)

R. Gouyon. *Le problème de mécanique rationnelle à l'Agrégation.* Pp. 256. 2000 fr. 1954. (Vuibert, Paris)

A. E. Green and W. Zerna. *Theoretical elasticity.* Pp. xiii, 442. 50s. 1954. (Geoffrey Cumberlege, Oxford University Press)

H. Griffiin. *Elementary theory of numbers.* Pp. ix, 203. 36s. 1954. (McGraw-Hill)

E. J. Gumbel. *Statistical theory of extreme values and some practical applications.* Pp. viii, 51. 40 cents. 1954. Applied mathematics series, 33. (National Bureau of Standards, Washington)

E. R. Hamilton and C. H. J. Smith. *Mathematics for living. III. Spending a holiday.* Limp, 4s. 6d. ; boards, 5s. 6d. 1954. (University of London Press)

I. Kaplansky. *Infinite abelian groups.* Pp. 91. $2. 1954. (University of Michigan Press, Ann Arbor)

M. Kline. *Mathematics in western culture.* Pp. xii, 484. 27 plates. 30s. 1954. (Allen & Unwin)

F. von Krbek. *Eingefangenes Unendlich.* Pp. iv, 332. DM 22. 1954. (Akademische Verlagsgesellschaft, Leipzig)

C. Kuratowski. *Topologie. II.* 2nd edition. Pp. viii, 443. $6. 1952. Monografje Matematyczne, 21. (Polska Akademia, Warsaw)

P. Levy. *Théorie de l'addition des variables aléatoires.* 2nd edition. Pp. xx, 387. 4500 fr. 1954. Monographies des probabilités, I. (Gauthier-Villars, Paris)

F. Mandl. *Quantum mechanics.* Pp. viii, 233. 35s. 1954. (Academic Press, New York ; Butterworth)

E. A. Maxwell. *An analytical calculus. III.* Pp. vii, 195. 15s. 1954. (Cambridge University Press)

J. Meixner and F. W. Schafke. *Mathieusche funktionen und Sphäroidfunktionen mit Anwendungen auf physicalische und technische Probleme.* Pp. xii, 414. DM 49 ; bound, DM 52.60. 1954. Grundlehren der mathematischen Wissenschaften, 71. (Springer, Berlin)

A. Monjallon. *Introduction à la méthode statisque.* Pp. 279. 2000 fr. 1954. (Vuibert, Paris)

C. G. Nobbs. *Elementary mathematics. I.* Pp. 336. 9s. 6d. 1954. (Geoffrey Cumberlege, Oxford University Press)

W. D. Reeve. *Mathematics for the secondary school.* Pp. xii, 547. $5.95. 1954. (Henry Holt, New York)

W. Haydn Richards. *Arithmetic made easy. III.* Parts 1 and 2. Pp. 102, 104. 3s. each. 1954. (Harrap)

C. H. Richardson. *An introduction to the calculus of finite differences.* Pp. vi, 142. 28s. 1954. (Van Nostrand, New York ; Macmillan)

R. Röthe. *Höhere mathematik für Mathematiker, Physider und Ingenieure. V. Formelsammlung.* 3rd edition. Pp. 124. DM 4.80. 1954. Mathematische Leitfäden, 43. (Teubner, Stuttgart)

G. Salmon. *A treatise on conic sections.* 6th edition., rep. Pp. xv, 399. Paper, $1.94 ; cloth, $3.25. 1954. (Chelsea Co., New York)

J. A. Schouten. *Tensor analysis for physicists.* 2nd edition. Pp. xii, 277. 30s. 1954. (Geoffrey Cumberlege, Oxford University Press)

W. Schmeidler. *Lineare Operatoren im Hilbertschen Raum.* Pp. 89. DM 7.80. 1954. Mathematische Leitfäden, 46. (Teubner, Stuttgart)

P. C. Sikkema. *Differential operators and differential equations of infinite order with constant coefficients.* Pp. 223. Fl. 11.50 ; cloth, fl. 13.50. 1953. (Noordhoff, Groningen)

C. A. B. Smith. *Biomathematics. The principles of mathematics for students of biological science.* 3rd edition, rewritten, of the work by the late W. M. Feldman. Pp. xv, 712. 80s. 1954. (Griffin)

E. R. Smith, S. Sielby and M. Kleiman. *Understanding college algebra.* Pp. xvi, 573. $3.50. 1954. (Dryden Press, New York)

E. S. Smith, M. Salkover and H. K. Justice. *Analytic geometry.* 2nd edition. Pp. xiii, 306. 32s. 1954. (John Wiley, New York ; Chapman & Hall)

L. Smith. *Exercises in workshop mathematics for young engineers.* Pp. vi, 90. 4s. 6d. 1954. (Cambridge University Press).

D. J. Struik. *A concise history of mathematics.* Pp. xix, 299. 14s. 1954. (Bell)

O. G. Sutton. *Mathematics in action.* Pp. viii, 226. 16s. 1954. (Bell)

K. Swainger. *Analysis of deformation. I. Mathematical theory.* Pp. xix, 285. 63s. 1954. (Chapman and Hall)

J. L. Synge. *Geometrical mechanics and de Broglie waves.* Pp. vii, 167. 25s. 1954. Cambridge monographs on mechanics and applied mathematics. (Cambridge University Press)

G. B. Thomas. *Calculus.* Pp. 614. $6.50. 1953. (Addison-Wesley, Cambridge, Mass.)

Sir Godfrey H. Thomson. *The geometry of mental measurement.* Pp. 60. 6s. 6d. 1954. (University of London Press)

W. J. Trjitzinsky. *Les problèmes de totalisation se rattachant aux Laplaciens non sommables.* Pp. 92. 1400 fr. 1954. Memorial des sciences mathématiques, 125. (Gauthier-Villars, Paris)

E. Trost. *Primzahlen.* Pp. 95. Sw. fr. 13.50. 1953. (Birkhäuser, Basel)

C. Truesdell. *The kinematics of vorticity.* Pp. 232. $6. 1954. (Indiana University Press, Bloomington)

J. D. Tucker and D. Wilkinson. *Radio. II.* Pp. ix, 252. 10s. 6d. 1954. (English Universities Press)

I. M. Vinogradov. *The method of trigonometrical sums in the theory of numbers.* Translated and revised by K. F. Roth and A. Davenport. Pp. x, 180. 33s. 1954. (Interscience Publishers)

I. M. Vinogradov. *Elements of number theory.* Translated from the 5th edition by S. Kravetz. Pp. viii, 227. $1.75 ; cloth, $3. 1954. (Dover, New York)

B. L. van der Waerden. *Science awakening.* Pp. 306. 1954. (Noordhoff, Groningen)

F. M. Warner. *Applied descriptive geometry with drafting-room problems.* 4th edition. Pp. viii, 247. 32s. 1954. (McGraw-Hill)

K. Yano and S. Bochner. *Curvature and Betti numbers.* Pp. ix, 189. 20s. 1953. *Annals of Mathematics studies*, 32. (Princeton University Press ; Geoffrey Cumberlege, London)

Colloque sur les fonctions de plusieurs variables. Pp. 164. 1500 fr. 1954. (Thone, Liège ; Masson, Paris)

Premier colloque sur les équations aux dérivées partielles. Pp. 132. 1400 fr. 1954. (Thone, Liège ; Masson, Paris)

Higher transcendental functions. I. II. Edited by A. Erdéiyi and the staff of the

Bateman Manuscript project. Pp. xxvi, 302; xvii, 396. 52s.; 60s. 1953. *Table of integral transforms. I.* Pp. xx, 391. 60s. 1954. (McGraw-Hill)

Tables of 10^x. Pp. 543. $3.50. 1953. *Tables of Lagrangian cofficients for sexagesimal interpolation.* Pp. 157. $2. 1954. *Tables of circular and hyperbolic sines and cosines for radian arguments.* Pp. x, 407. $3. 1953. *Tables of secants and cosecants to nine significant figures at hundredths of a degree.* Pp. vi, 46. 35s. 1954. Applied mathematics series, 27, 35, 36, 40. (National Bureau of Standards, Washington)

Table of binomial coefficients. Edited by J. C. P. Miller. Pp. viii, 162. 35s. 1954. Royal Society Mathematical Tables, 3. (Cambridge University Press)

A short table of Bessel functions of integer orders and large arguments. Prepared by L. Fox. Pp. 28. 6s. 6d. 1954. Royal Society Shorter Mathematical Tables, 3. (Cambridge University Press)

A selection of graphs for use in calculations of compressible airflow. Prepared on behalf of the Aeronautical Research Council. Pp. x, 115. 84s. 1954. (Geoffrey Cumberlege, Oxford University Press)

Enzyklopädie der mathematischen Wissenschaften. I_2, 11, 3. *Geometrie der Zahlen.* Prepared by O. H. Keller. DM 8.80. 1954. (Teubner, Leipzig)

The geometry of René Descartes. Translated by D. E. Smith and M. L. Latham. Pp. xiii, 244. $1.50; cloth, $2.95. 1954. (Dover, New York)

Selected papers on noise and stochastic processes. Pp. 370. $2; cloth, $3.50. 1954. (Dover, New York)

Wave motion and vibration theory. Pp. v, 169. 56s. 1954. Symposia in applied mathematics, 5. (McGraw-Hill)

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